# On the location of public bads: strategy-proofness under two-dimensional single-dipped preferences

Murat Öztürk[†] Hans Peters[*] Ton Storcken[*]

[†]MEF University
Department of Economics
[*]Maastricht University
Department of Quantitative Economics

Conference on Economic Design
SED 2015, Turkey

# Overview of the presentation

# Example

## Example

A public bad, eg. nuclear plant, has to be located in some specific area.
Each person regards one position (the "dip") for this public bad as the worst and would like the public bad to be located as far away (Euclidean distance) from this dip as possible.
The dip is private knowledge.

## Question

Is there a strategy-proof and Pareto optimal rule that can be used to determine the location of the public bad?

# Basic Model

- $N = \{1, \ldots, n\}$ ($n \geq 2$) is the set of *agents*
- $A \subseteq \mathbb{R}^m$ ($m \geq 2$) is set of *alternatives*
- Each $a \in A$ can be identified with a (*single-dipped*) *preference* $R^a$: for $x, y \in A$, $xR^a y \Leftrightarrow ||x - a|| \geq ||y - a||$, where $||x - a||$ denotes Euclidean distance between $x$ and $a$
- A *solution* assigns to a profile of preferences a point of $A$: $f((a^i)_{i \in N}) \in A$, where $a^i \in A$ for all $i \in N$
- $f$ is *Pareto optimal* (PO) if for every profile $(a^i)_{i \in N}$ there is no $a \in A$ such that $||a - a^j|| \geq ||f((a^i)_{i \in N}) - a^j||$ for all $j \in N$ with at least one inequality strict

## Basic Model

- $f$ is *strategy-proof* (SP) if for every profile $(a^i)_{i \in N}$, every $j \in N$ and every $b^j \in A$, we have:

$$||a^j - f((a^i)_{i \in N})|| \geq ||a^j - f((a^i)_{i \neq j}, b^j)||$$

- $f$ is *intermediate strategy-proof* (ISP) if for every $S \subseteq N$, every profile $(a^i)_{i \in N}$ with $a^j = a^k$ for all $j, k \in S$ and every $(b^i)_{i \in S}$, we have:

$$||a^S - f((a^i)_{i \in N})|| \geq ||a^S - f((a^i)_{i \notin S}, (b^i)_{i \in S})||$$

**Lemma**

*$f$ is strategy-proof $\iff$ $f$ is intermediate strategy-proof*

# Literature

*Public bads,* $m = 1$

- Peremans and Storcken (1999)
  characterization of strategy-proof rules
- Barberà, Berga, and Moreno (2009)
- Manjunath (2009)
  more explicit results under more restrictions
- Ehlers (2002)
  probabilistic rules with with single-dipped preferences

# Literature

*Private bads, $m = 1$*

- Klaus, Peters, and Storcken (1997)
  private bads, single-dipped version of Sprumont (1991)

*Further*

- Literature on ‘value-restrictions’, which include single-dippedness.
- Inada (1964), Sen and Pattanaik (1969)

*In our work*

- $m = 2$, $A$ is a convex polygon including its interior.
- $m = 2$, $A$ is a disc.

# Convex Polygons

We will consider $A$ as convex polygons with their interiors. First some remarks and a general result.

- Mainly we will consider profiles with two dips: there exists a coalition $S \subseteq N$ and points $a, b \in A$ such that all agents in $S$ have point $a$ as their dip and all agents in $N \backslash S$ have point $b$ as their dip.
- Notation: $p = (a^S, b^{N \backslash S})$ will be called a conflict.

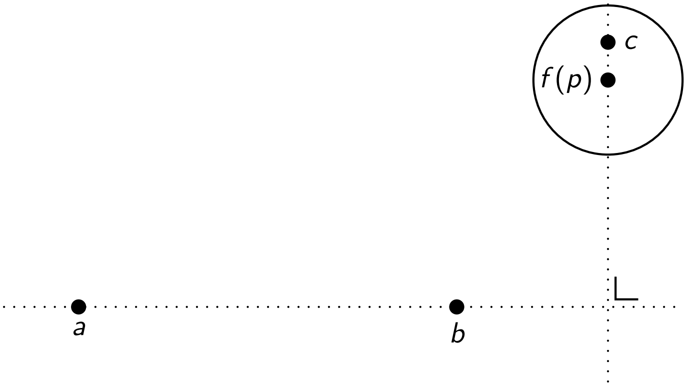
**Lemma: PO implies Outcome is on Boundary**

Let $A \subseteq \mathbb{R}^m$ with $m \geq 2$, $p = (a^S, b^{N \backslash S})$ be a conflict and $f$ be a Pareto optimal solution. Then $f(p)$ is a boundary point of $A$.

# Convex Polygons

## Proof of Lemma

Suppose, to the contrary, that $f(p)$ is an interior point of $A$, i.e., there is an open ball around $f(p)$ contained in $A$.



# Convex Polygons

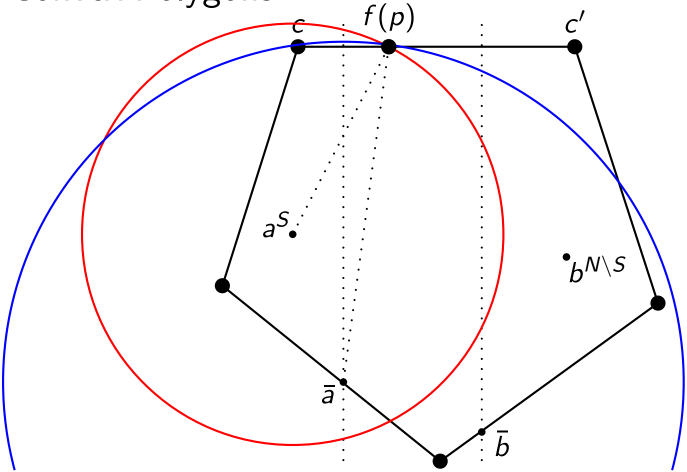
## Lemma: Outcome is on Boundary implies Outcome is Vertex

Let $A \subseteq \mathbb{R}^2$ be a convex polygon with its interior, $p$ be a single-dipped preference profile, $f$ be Pareto optimal and strategy-proof solution, and $f(p)$ be on boundary of $A$. Then $f(p)$ is a vertex point of $A$.

**Demonstration of Proof**

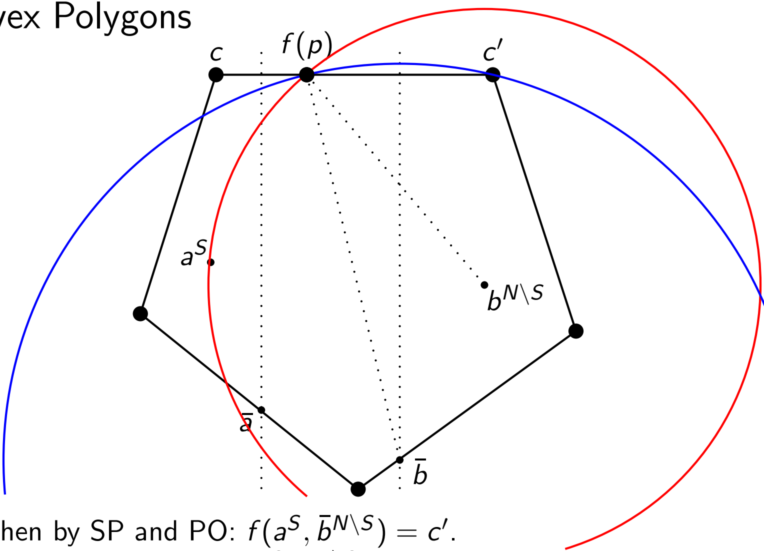
Let $A$ be a pentagon (regular polygon with 5 edges) with its interior, and $p = (a^S, b^{N \setminus S})$ be a conflict.

# Convex Polygons




Then by SP and PO: $f(\bar{a}^S, b^{N\setminus S}) = c$.
Hence again by SP: $f(\bar{a}^S, \bar{b}^{N\setminus S}) = c$.

# Convex Polygons

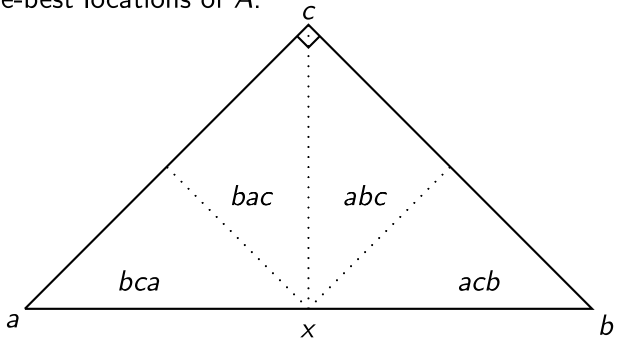



Then by SP and PO: $f(a^S, \bar{b}^{N\setminus S}) = c'$.
Hence again by SP: $f(\bar{a}^S, \bar{b}^{N\setminus S}) = c'$.

# Convex Polygons

## Single-Best Locations

- Notation: $best(a) = \{b \in A \mid d(a,b) \geq d(a,c) \text{ for all } c \in A\}$.
- Notation: $\mathcal{B} = \{a \in A \mid best(x) = \{a\} \text{ for some } x \in A\}$ is set of single-best locations of $A$.




- $best(x) = \{a, b, c\}$, but $\mathcal{B} = \{a, b\}$.

# Convex Polygons

## Lemma: Two Dips Profiles

Let $A$ be a convex polygon with its interior, $S \subseteq N$, $f$ be a Pareto optimal and strategy-proof solution, $p = (x^S, y^{N \setminus S})$ with $a = best(x)$, $b = best(y)$ and $a, b \in \mathcal{B}$. Then $f(p) \in \{a, b\}$.

# Convex Polygons

- Notation: $S \subseteq N$ is *decisive* if all agents in coalition $S$ have some point $a \in A$ as their dip in a single-dipped preference profile $p$ then $f(p) \in best(a)$.
- $|\mathcal{B}| \geq 3$. The case $|\mathcal{B}| = 2$ will be investigated later.

**Lemma: Decisive Coalitions**

Let $A$ be a convex polygon with its interior, $S \subseteq N$ and $f$ be a Pareto optimal and strategy-proof solution. Then either $S$ or $N \backslash S$ is decisive.

# Convex Polygons

## Lemma: Intersection of Decisive Coalitions

Let the following conditions hold:
(i) $|\mathcal{B}| \geq 3$
(ii) there are distinct $a, b, c \in \mathcal{B}$, and there is no $x \in \mathcal{B}$ such that $\{a, b, c, x\}$ is rectangular
Let $A$ be a convex polygon with its interior, $S \subseteq N$ and $f$ be a Pareto optimal and strategy-proof solution. If $S$ and $T$ are both decisive then $S \cap T$ is decisive.

# Convex Polygons

## Lemma: Set of Decisive Coalitions is Ultrafilter

Let $A$ be a convex polygon with its interior, $f$ be Pareto optimal and strategy-proof. Then the set of decisive coalitions is an ultrafilter $\mathcal{F}$

- $\emptyset \notin \mathcal{F}$
- if $S, T \in \mathcal{F}$, then $S \cap T \in \mathcal{F}$ for all $S, T \subseteq N$
- $S \in \mathcal{F}$ or $N \backslash S \in \mathcal{F}$ for all $S \subseteq N$.

## Consequence

There is a unique $d \in N$ with $\{d\} \in \mathcal{F}$.

## Proof

Otherwise $N \backslash \{i\}$ is decisive for all $i \in N$. So, $\cap \{N \backslash \{i\} : i \in N\} = \emptyset$ is decisive. □

# Convex Polygons

## Definition: Dictatorial

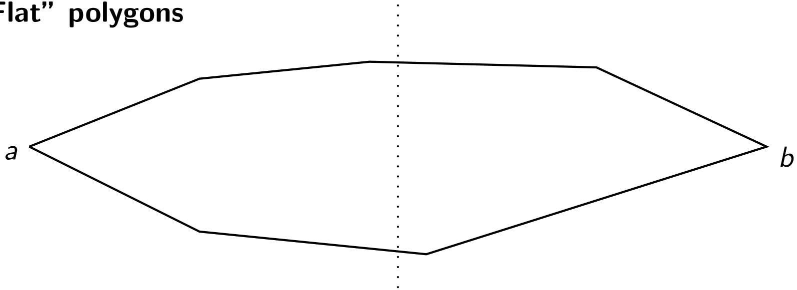
Solution $f$ is dictatorial if there exists a $d \in N$, called the dictator, such that for every profile $p$ we have $f(p) \in best(a)$, where $a$ is the dip of $d$.

## Theorem: Dictatorship on Convex Polygons

Let $A$ be a convex polygon with its interior, except cases $|\mathcal{B}| = 2$ and $\mathcal{B}$ is rectangular, and $f$ be a Pareto optimal and strategy-proof solution. Then $f$ is dictatorial, i.e. there is an agent $i \in N$ such that $f(p) \in best(p(i))$ for all $p$.

# Convex Polygons: $|\mathcal{B}| = 2$

**"Flat" polygons**




- $\mathcal{W}_a^f = \{(S, U) \in N \times N \mid S \cap U = \emptyset,\ f(p) = a$ for all profiles $p$ with $a$ is single best for all agents in $S$ and both $a$ and $b$ are best for all agents in $U\}$.
- $\mathcal{W}_b^f = \{(T, U) \in N \times N \mid T \cap U = \emptyset,\ f(p) = b$ for all profiles $p$ with $b$ is single best for all agents in $T$ and both $a$ and $b$ are best for all agents in $U\}$.

# Convex Polygons: $|\mathcal{B}| = 2$

The pair $(\mathcal{W}_a^f, \mathcal{W}_b^f)$ is

- *proper and strong*: either $(S, U) \in \mathcal{W}_a^f$ or $(T, U) \in \mathcal{W}_b^f$ for all pairwise disjoint sets $S, U$ and $T$ with $S \cup T \cup U = N$,
- *Pareto optimal*: $(S, U) \in \mathcal{W}_a^f$ in case $S \cup U = N$, and $(T, U) \in \mathcal{W}_b^f$ in case $T \cup U = N$,
- *monotone*: $(S', U') \in \mathcal{W}_a^f$ whenever $(S, U) \in \mathcal{W}_a^f$, $S \subseteq S'$ and $S \cup U \subseteq S' \cup U'$ and $(T', U') \in \mathcal{W}_b^f$ whenever $(T, U) \in \mathcal{W}_b^f$, $T \subseteq T'$ and $T \cup U \subseteq T' \cup U'$.

# Convex Polygons: $|\mathcal{B}| = 2$

## Definition

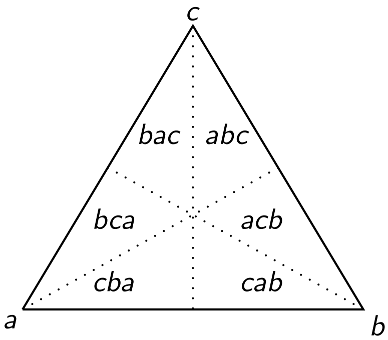
f is *non-corruptive*, if $f(p) = f(q)$ for any profiles $p, q$ with $p(j) = q(j)$ for all $j \in N \setminus \{i\}$, $||p(i) - f(p)|| = ||p(i) - f(q)||$ and $||q(i) - f(p)|| = ||q(i) - f(q)||$.

## Lemma

For any proper, strong, Pareto optimal and monotone pair $(\mathcal{W}_a, \mathcal{W}_b)$, there is a Pareto optimal, strategy-proof and non-corrupt solution $f$ such that $\mathcal{W}_a = \mathcal{W}_a^f$ and $\mathcal{W}_b = \mathcal{W}_b^f$.

# Convex Polygons: Examples

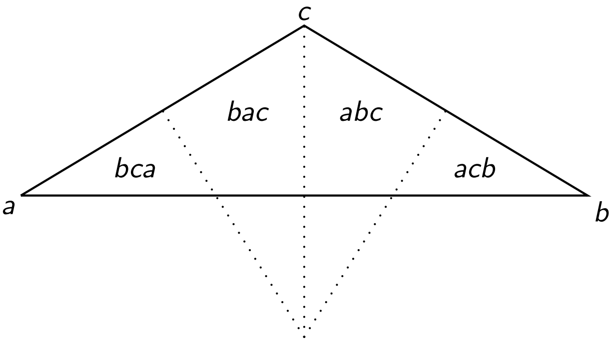
**Equilateral triangles**




- For a equilateral triangle, domain over vertices is full
- Impossibility result

# Convex Polygons: Examples

**Flat triangles**

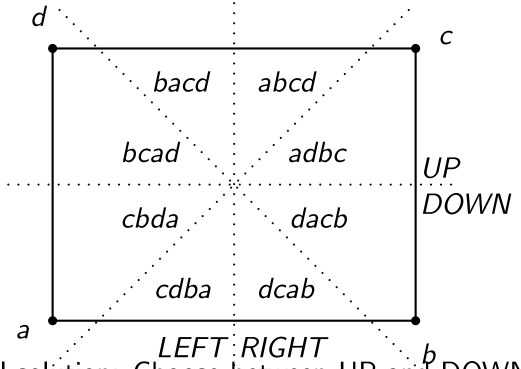



- For a "flat" triangle, domain over vertices is not full
- Non-dictatorial solution: majority voting between $a$ and $b$

# Convex Polygons: $|\mathcal{B}| = 4$

**Rectangular case**

- $\mathcal{B} = 4$ and $\mathcal{B}$ equals to set of four corner points of a rectangle




Non-dictatorial solution: Choose between UP and DOWN by majority voting, choose between LEFT and RIGHT by majority voting.

# Disc

**Theorem: Dictatorship on Disc**

Let $A$ be a disc, and $f$ be a Pareto optimal and strategy-proof solution. Then $f$ is dictatorial, i.e. there is an agent $i$ with dip $a$ such that $f(p) \in best(a)$ for all single-dipped preference profile $p$.

- Intuitively, this result was expected.
- But we did not manage to prove this theorem from the theorem for convex polygons.
- The proof is quite similar to the proof(s) for convex polygons.

# Summary and Open Questions

## Summary

- Let $A \subseteq \mathbb{R}^m$ $(m \geq 2)$ be compact, and let $f$ be a strategy-proof and Pareto optimal solution for single-dipped preference profiles. Then $f$ always assigns a boundary point of $A$.
- If $m = 2$ and boundary of $A$ is a convex polygon then $f$ is dictatorial except two cases.
- For these two cases strategy-proof, Pareto optimal and non-corruptive solutions are characterized.
- If $m = 2$ and $A$ is a disc, then $f$ is dictatorial.

## Open Questions

- General convex sets?
- More than two dimensions? $(m \geq 3)$

THANK YOU FOR YOUR ATTENTION