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Teachers' Attention to and Flexibility with Referent Units

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Abstract
Attending to the whole unit that a number refers to in a mathematical problem situation and showing flexibility in coordinating different units are foundational for mathematical understanding. In this study, we explored teachers' attention to and flexibility with referent units in situations involving fractions and fraction multiplication. Using data collected across the USA from 246 mathematics teachers in Grades 3–7 where fractions are taught, we found that teachers' attention to and flexibility with referent units were related to each other as well as to teachers' overall knowledge of fractions.

Keywords Fractions · Mathematical knowledge for teaching · Mathematics education · Referent units · Teacher knowledge

Many would agree that teachers should have a deep understanding of the concepts they are expected to teach. Indeed, various documents have underscored the importance of teachers having a robust understanding of school mathematics for teaching and student learning (National Council of Teachers of Mathematics [NCTM], 2000; National Mathematics Advisory Panel, 2008). Prior work has supported empirical evidence on the critical role teachers' mathematical knowledge played in the quality of mathematics instruction (e.g., Copur-Gencturk, 2015; Borko et al., 1992; Tchoshanov, 2011) and students' mathematics learning (e.g., Baumert et al., 2010; Hill, Rowan, & Ball, 2005; Kersting, Givvin, Thompson, Santagata, & Stigler, 2012). Given that fractions make up a significant portion of the mathematics concepts introduced in the upper elementary and middle school grades (e.g., Common Core State Standard Initiatives [CCSSI], 2010) and are necessary for further study in mathematics (Booth, Newton, & Twiss-Garrity, 2014; Hackenberg & Lee, 2015; Siegler et al., 2012), investigating teachers' understanding of fractions is warranted.

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Indeed, numerous studies have focused on in-service and future teachers' understanding of fractions. The overall findings of these studies indicate that U.S. teachers have a partial conceptual understanding of fractions (e.g., Armstrong & Bezuk, 1995; Ball, Lubienski, & Mewborn, 2001; Bradshaw, Izsák, Templin, & Jacobson, 2014; Copur-Gencturk, 2021b; Hohensee & Jansen, 2017; Izsák, 2008; Jansen & Hohensee, 2016; Lee, 2017; Ma, 1999; Newton, 2008; Simon, 1993). More recent work suggests that identifying teachers' understanding of the wholes fractions refer to could shed light on the depth of teachers' understanding of fractions and fraction operations as well as the learning environment they create for their students (e.g., Izsák, 2008; Izsák, Jacobson, & Bradshaw, 2019; Lee, 2017; Lee, Brown, & Orrill, 2011; Lo & Luo, 2012; Philipp & Hawthorne, 2015; Simon, 1993).

Yet the majority of these studies have focused on teachers' *flexibility with referent units*, which can be defined as "a teacher's ability to keep track of the unit to which a fraction refers . . . and to shift their relative understanding . . . as the referent unit changes" (Lee et al., 2011, p. 204). In the present study, in addition to examining teachers' flexibility with referent units, as in many prior works, we investigated teachers' attention to the referent units, given that noticing is an important component of how a person constructs "the whole" (von Glasersfeld, 1981). We also examined how teachers' understanding of referent wholes from these two aspects (i.e., attention to and flexibility with referent units) is related to each other as well as to teachers' overall understanding of fractions.

In the following section, we provide theoretical justifications for why we focused on these two aspects of teachers' understanding of referent units. We then review research related to teachers' understanding of referent units, followed by a description of the study context and the research method. We close the paper by discussing the implications of the study for future teacher education and preparation programs.

Conceptual Framework

Grasping number relations is a rational process that involves "conceiving a whole of parts and relating parts in a definite whole" (McLellan & Dewey, 1895, p. 30–31). Thus, the construction of a unit, what is "one" whole, plays a critical role in developing children's number sense (von Glasersfeld, 1981). As Behr and colleagues (1997) wrote, "The nature of the unit that a person uses and transforms as the argument of fractioning procedure is of central importance in attempting to describe and model his or her concept of rational number and operations with rational numbers" (p. 49).

The construction of a unit is related to the patterns a person attends to. As von Glasersfeld (1981) pointed out,

> unitizing operations consist in the differential distribution of focused and unfocused attentional pulses. A group of co-occurring sensory-motor signals becomes a "whole" or a "thing" or "object" when an unbroken sequence is framed or bonded by an unfocused pulse at both ends. (p. 87).

Thus, attending to the unit that numbers are referring to is an important aspect of what mental representation a person uses to construct "the whole." For this research, we investigated teachers' attention to the units from a reverse order: by asking them to decide and explain whether it is possible for 1/3 to be greater than 1/2. In this situation, fractions were represented without unit wholes, so teachers' responses could provide evidence showing whether they noticed the units to which these numbers could refer.

Another important aspect of understanding the unit concept is the coordination of units. Indeed, once a unit is identified, new levels of units can be coordinated by taking these identified units and constructing new units that comprise them (von Glasersfeld, 1981). Unit iterations and coordinating the various levels of units are foundational for understanding several mathematics concepts, such as fractions, the base-10 place value system, and operations. A growing body of research suggests that the number of levels of units that students or teachers can coordinate is critical for conceptualizing fractions, fraction operations, and other mathematical concepts, such as calculus (e.g., Hackenberg & Tillema, 2009; Hunting, 1983; Steffe & Olive, 2010). For this research, we captured teachers' flexibility with referent units by investigating how teachers coordinated different referent units for fractions involved in a multiplication problem. Specifically, teachers were given a drawn rectangle with the multiplicand represented. They were then asked to finish modeling the fraction multiplication, which required them to coordinate different referent wholes for the multiplier and the product, thereby showing flexibility with the referent units.

Let us illustrate why showing flexibility with referent units is important for the learning environment teachers create for their students. The correct representation of a multiplication problem requires coordinating different referent units to accurately represent the information in a given problem. The following multiplication word problem can be modeled by using a drawn rectangle, as in our study: "One serving of yogurt is 1/3 of a cup. For one meal, Amanda ate 1/2 of a serving. How many cups of yogurt did Amanda eat?" One serving size of yogurt (i.e., the multiplicand) can be represented by shading 1/3 of the rectangle, which represents a cup (see Fig. 1a). Because Amanda ate 1/2 of the serving, to model the problem accurately, 1/2 of *the serving size* (i.e., 1/3 of a cup) should be shaded (see Fig. 1b). Finally, given that the question asks how many *cups* of yogurt Amanda ate, the referent unit of the product 1/6 should become the original whole rectangle (Fig. 1c).

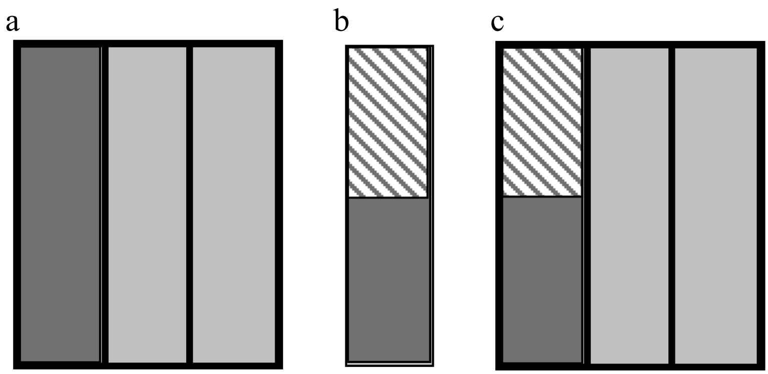


Fig. 1 Flexibility with referent units. **a** One-third of the whole rectangle represents the serving size as 1/3 of a cup; **b** One-half of the partial rectangle represents 1/2 of the *serving size*; **c** One-sixth of the whole rectangle represents 1/6 of a cup

To further illustrate the importance of showing flexibility with referent units, let us contrast this strategy with a method commonly used by teachers to model the same multiplication problem using drawn rectangles: the overlapping method. In this method, teachers usually shade the multiplier and multiplicand horizontally and vertically (or vice versa) and then identify the overlapped shaded area to represent the product. Thus, in this method of modeling, the referent wholes for the multiplier (1/2), multiplicand (1/3), and the product (1/6) are represented by the same rectangle (see Fig. 2). Notice that this way of representing the situation does not match the problem because the serving size is 1/3 of a cup (i.e., the full rectangle) and Amanda ate 1/2 of *the serving size* (not the full cup represented by the whole rectangle). We consider the overlapping method, as used in this example, problematic for teaching because we contend that "there is no reliable way to go from a problem statement to a solution procedure unless you get a correct representation of the problem" (Davis & Maher, 1990, p. 75). We argue that this representation is not correctly mapping the given problem; rather, it mimics the step-by-step algorithm (e.g., Lee et al., 2011; Webel, Krupa, & McManus, 2016).

We argue that focusing on teachers' understanding of the two aspects of referent units (i.e., attention and flexibility) would provide further insights into the learning environment teachers create for their students to understand fraction concepts. A conceptual understanding of fractions requires one to attend explicitly to the units and to be aware of different units in multiplication and division situations (Philipp & Hawthorne, 2015). Additionally, a growing body of research suggests that students' ability to coordinate more levels of units is critical for their ability to conceptualize fractions and fraction operations and to understand mathematical concepts (e.g., Hackenberg & Tillema, 2009; Steffe & Olive, 2010). Thus, we expected that teachers' understanding of referent units would be associated with their overall understanding of fractions.

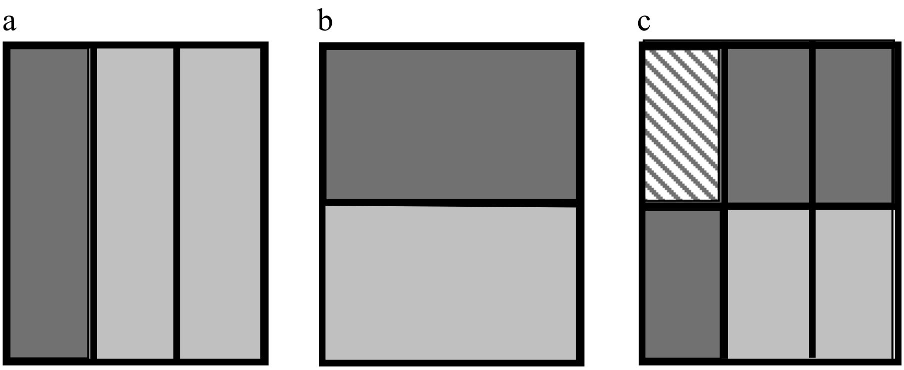


Fig. 2 Visual representation of the overlapping method. **a** The full rectangle represents one cup and one-third of the full rectangle represents the serving size; **b** the full rectangle represents the serving size; **c** the full rectangle represents one cup

Prior Research on Referent Units

Prior work has focused heavily on teachers' understanding of referent units in situations where the referent units change during the process, such as in modeling fraction multiplication and division situations (e.g., Izsák, Orrill, Cohen, & Brown, 2010; Lee, 2017; Lee et al., 2011; Webel et al., 2016). The findings across these studies have been similar in that both in-service and future teachers have struggled to identify or coordinate referent wholes in mathematical problems (e.g., Baek et al., 2017; Izsák, 2008; Lee, 2017; Lee et al., 2011; Luo, Lo, & Leu, 2011; Son & Lee, 2016; Tobias, 2013; Webel et al., 2016; Webel & DeLeeuw, 2016; Whitehead & Walkowiak, 2017). For instance, future teachers seem to struggle with writing word problems for fraction arithmetic because they cannot identify the whole units to which each fraction refers (e.g., McAllister & Beaver, 2012; Toluk-Uçar, 2009).

Teachers' reliance on the overlapping method to model fraction multiplication has also been documented in prior studies (e.g., Izsák, 2008; Webel et al., 2016; Webel & DeLeeuw, 2016). Lee (2017), who investigated 111 future U.S. elementary teachers' written solutions to a measurement fraction division problem using a given length model, found that teachers' struggles with coordinating referent units flexibly were not specific to multiplication situations. Lee (2017) found that only 13 teachers were able to use drawings to show the corresponding referent units for the divisor, dividend, and quotient. Other teachers' use of drawings indicated that even those who answered the problem correctly could not show the process of division visually. In a more recent study, Izsák et al. (2019) investigated teachers' reasoning about fraction arithmetic by collecting data from a national sample of 990 U.S. in-service middle-grade teachers. The authors found that teachers' identification of and flexibility with referent units were partial. In fact, only 30% of the participants showed mastery of referent units.

The role that teachers' understanding of referent units plays in their overall understanding of fractions is not clear. For instance, Ölmez and Izsák (2020) further analyzed the same data to identify latent classes of teachers and found that teachers' flexibility with referent units for fraction multiplication or division was not a distinguishing characteristic of the three latent classes of teachers they identified. Yet, in Izsák et al.' (2010) prior work based on the responses of a sample of 201 U.S. middle-grade teachers to a set of multiple-choice items on rational number arithmetic, they found that teachers' understanding of referent units was a distinguishing characteristic of the two groups identified. Taken altogether, we argue that the role of teachers' understanding of certain aspects of referent units in their overall understanding of fractions warrants further investigation.

As noted above, past research has focused heavily on teachers' flexibility with referent units and has noted teachers' struggles with referent units. In most of these studies, teachers' understanding of referent units was examined by how they coordinated different units in given drawings or real-world situations (i.e., flexibility with referent units, Izsák et al., 2019; Lee, 2017). Although using drawings and real-world situations allows researchers to examine teachers' understanding of changing referent units during an operation, it may not provide much insight into teachers' attention to referent units. This contention is supported by the results of a study conducted by Bartell, Webel, Bowen, and Dyson (2013) in which they investigated preservice teachers' learning to recognize children's mathematical understanding from an intervention designed for this purpose. Bartell et al. (2013) collected data from 54 future elementary teachers before the intervention to capture their content knowledge and its role in supporting their analysis of children's understanding. In one of the content knowledge items, participants were asked to circle the larger fraction, 5/6 or 6/7, and explain their thinking. The authors noted that 27% of the future teachers used different referent units for 5/6 and 6/7. This finding also warrants further investigation of how teachers attend to referent units when no referent unit is provided for them.

In the present study, we aimed to address this gap in the current literature by studying teachers' attention to and flexibility with referent units and how the two aspects are related to one another. Furthermore, we anticipated that teachers who demonstrated an understanding of these two aspects of referent units would also show a more robust understanding of fraction concepts in problems that were not capturing their knowledge of referent units. From data collected from 246 U.S. in-service teachers who were teaching mathematics in grades where fraction concepts are taught, we aimed to answer the following research questions:

1. To what extent do teachers demonstrate attention to referent units?
2. To what extent do teachers demonstrate flexibility with referent units?
3. What is the relationship between teachers' attention to and flexibility with referent units?
4. To what extent are teachers' attention to and flexibility with referent units associated with their overall understanding of fractions?

Methods

Participants

The data used in this study were collected from 246 in-service mathematics teachers in Grades 3–7 (i.e., children ages 8 to 13) across 21 states in the USA. We identified these teachers through a district with which we partner and an educational research company that provided the contact information of teachers. Teachers received an invitation to participate in the study and were compensated for their participation by receiving an online gift card. We developed an online survey in which teachers provided information regarding their educational background and then answered a set of mathematics tasks that required them to type in their responses. The mathematics tasks were presented in a randomized order, and the teachers were not allowed to continue until they had provided a response to each question. We asked the teachers to write a response to share their thinking even if they were not sure how to solve a problem or even if they did not know the answer to a specific problem. In this way, we were able to capture their thinking on all the tasks.

Table 1 presents the demographics and professional backgrounds of the teachers in the study sample. Most of the teachers in our sample were female (84%) and White (68.1%). Additionally, 77% of the teachers were teaching mathematics in Grades 3–5.

Table 1 Teachers' demographics and professional background characteristics

Variable	Our sample (%)
Teacher background	
Gender (female)	84.0
Ethnicity (White)	68.1
Master's degree (yes)	25.2
Teaching level	
Elementary school (Grades 3–5)	76.4
Middle school (Grades 6 & 7)	23.6
Professional background	
Traditional certification	70.6
Credential in mathematics	19.3
Fully certified	52.5

Note. $N = 238$

Referent Unit Tasks

We measured teachers' attention to referent units by using a task adapted from a teacher education resource (Van de Walle, Karp, & Bay-Williams, 2019). Teachers' responses to the question of whether it would be possible for 1/3 to be greater than 1/2 were used as evidence for their attention to referent units (see Fig. 3). We anticipated that teachers who were attending to referent units would include a referent unit to justify their answer. We captured their flexibility with referent units by using a question taken from a validated assessment (Izsák et al., 2019). Teachers were provided with a drawn rectangle that showed the referent unit for 1/4. Thus, we anticipated that teachers who showed flexibility with referent units would use 1/4 as the referent unit for 1/3 and the whole rectangle for 1/12.

Attention to referent units	Is it possible for $\frac{1}{3}$ to be greater than $\frac{1}{2}$? Explain your thinking.
Flexibility with referent units	A middle school teacher, Ms. Bender, wants to model what the statement $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$ means conceptually. To model the problem, she starts by drawing a rectangle to show the one whole. Next, she divides the rectangle horizontally into 4 equal-sized parts and shades in one part.

Explain the remaining steps Ms. Bender should follow to model that the product of $\frac{1}{3} \times \frac{1}{4}$ is $\frac{1}{12}$.

Fig. 3 Referent unit tasks

Coding of Teachers' Responses to Tasks on Attention to and Flexibility with Referent Units

We followed an iterative process to code teachers' responses to the two tasks. Specifically, we identified categories based on the existing literature and then added or refined those categories based on our initial coding of a subsample of the data. During this process, both authors coded the data together for training purposes and discussed our coding. After we had established an understanding of each category, we independently coded a random sample of 20 responses for each task. The agreement was 92% and 95% for the tasks on attention to referent units and flexibility with referent units, respectively. We discussed the codes on which we disagreed and reached an agreement. We coded the remaining teachers' responses independently, then compared our codes and again discussed any disagreements.

We classified teachers' responses to the *attention to referent units* task into three groups. The teachers whose responses fell into Group 1 did not explicitly use a referent unit to justify their responses. For instance, one teacher whose response fell into Group 1 said, "We have to have common denominators. 1/3 = 2/6 and 1/2 = 3/6. Therefore, 2/6 is less than 3/6." As shown in this teacher's response, attention was not explicitly given to the referent unit to which 1/3 and 1/2 were referring. Responses in Group 2 included teachers who said that 1/3 could not be greater than 1/2 and used the same referent unit to justify their responses. For instance, one teacher wrote:

> No, 1/3 is equal to 33%, whereas 1/2 is equal to 50%. If you look at the number 10, 1/3 of that is roughly 3, and half of it is 5. If you use the number 100, 1/3 of that is about 33, and 1/2 of it is 50.

The teachers whose responses fell into Group 3 responded that the answer depended on the referent unit. These teachers' explanations provided evidence that they had attended to the referent units. For instance, one teacher explained, "If it is 1/3 or 1/2 of the same amount, then no. One third of a gallon would be bigger than 1/2 of a cup, but those would not be 1/3 or 1/2 of the same thing."

We used a similar approach to categorize teachers' responses to the *flexibility with referent units* task. Specifically, we also created three groups. The responses in Group 1 included those who stated they did not know how to model it or who did not explicitly report referent units for the multiplier 1/3 or the product 1/12. Most of the responses in Group 1 were, "I am unsure how to model that the product of 1/3 × 1/4 is 1/12. I don't see how this problem can be modeled visually," or "She should draw 3 vertical lines dividing the rectangle into 12 equal sections." As in the second sample response from Group 1, it is not clear what referent whole this particular teacher was using in his or her modeling.

The teachers whose responses fell into Group 2 used the overlapping method, meaning that they used the same referent unit for the multiplier, multiplicand, and product (e.g., Izsák, 2008; Webel et al., 2016; Webel & DeLeeuw, 2016). Group 2 teachers generally represented each fraction by adding vertical lines and then choosing the double-shaded or overlapping area. As an example, one teacher in Group 2 wrote, "She should draw two vertical lines to divide the rectangle into 3 equal-sized parts across, then shade in one of the vertical rectangles. The shaded piece that is overlapped demonstrates the 1/12."

The final category (i.e., Group 3) included responses in which teachers coordinated different referent units for the multiplier and the product. The teachers whose responses fell into Group 3 used 1/4 of the entire rectangle as a referent unit for 1/3 and used the whole rectangle as the referent unit for the product (i.e., 1/12). A sample response from Group 3 is, "She should divide the picture into 3 equal-sized pieces vertically and show that 1/3 of the 1/4 is 1/12 of the whole."

Fraction Measure

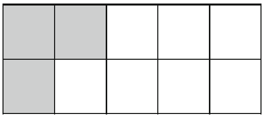
To investigate the role of teachers' knowledge of referent units in their overall understanding of fraction concepts, we created a measure consisting of tasks that did not require teachers to draw on their attention to or flexibility with referent units directly. We hypothesized if teachers' understanding of referent units is foundational for their overall knowledge of fractions, we would expect that teachers whose responses demonstrated attention to and flexibility with referent units would perform better on this measure than those whose responses did not demonstrate attention to or flexibility with referent units.

We created a fraction measure by adapting tasks from prior research (e.g., Siegler & Lortie-Forgues, 2015) and existing teacher knowledge assessments (Tatto et al., 2012). Specifically, the first two tasks were designed to capture teachers' understanding of equivalent fractions and the ability to compare fractions (Tatto et al., 2012), whereas the last two tasks were designed to measure teachers' conceptual understanding of fraction operations (e.g., Siegler & Lortie-Forgues, 2015). As shown in Table 2, teachers were asked to locate their estimations on a given number line and to explain the strategy they used to estimate the sum or quotient (see Copur-Gencturk, 2021a, for further details). We tested the validity of these tasks by conducting interviews with 20 teachers. The first author and another rater also coded the teachers' explanations separately. The raters' agreement on individual items was greater than 90%, and disagreements were resolved by discussion. Teachers' scores on these items were then converted to Z-scores. Cronbach's alpha (an indicator of the reliability of a measure) was .72, indicating good reliability. The mean score on this measure was 0, with a standard deviation of .49.

Teacher Background Characteristics

We created a set of indicators to assess the role of teachers' professional background in the observed relationship between their understanding of referent units and their performance on the fraction measure. Specifically, binary variables were created for the certification type (with traditionally certified being the reference category) and being middle school mathematics teachers (with being an upper elementary school mathematics teacher as the reference category).

Table 2 Fraction measure tasks

<table>
<tr><th>Key concept</th><th>Task</th></tr>
<tr><td>Equivalent fractions</td><td>In the figure, how many MORE small squares need to be shaded so that 4/5 of the total number of small squares are shaded? Explain your answer.
</td></tr>
<tr><td>Comparing fractions</td><td>For each set of fractions, put < , >, or = to make the statement true.
$\frac{9}{21}$ $\frac{15}{21}$ $\frac{31}{57}$ $\frac{23}{57}$ $\frac{61}{44}$ $\frac{63}{44}$
$\frac{20}{17}$ $\frac{20}{33}$ $\frac{49}{48}$ $\frac{49}{47}$ $\frac{71}{60}$ $\frac{71}{52}$</td></tr>
<tr><td>Estimating the sum of fractions</td><td>The fractions $\frac{19}{35}$ and $\frac{41}{66}$ have been placed on a number line. Without computing, please estimate the sum of $\frac{19}{35}+\frac{41}{66}$ by placing a dot on the number line where you think the sum would be found. Explain your answer.

</td></tr>
<tr><td>Estimating the quotient of fractions</td><td>The fractions $\frac{19}{35}$ and $\frac{41}{66}$ have been placed on a number line. Without computing, please estimate the quotient of $\frac{41}{66}\div\frac{19}{35}$ by placing a dot on the number line where you think the quotient would be found. Explain your answer.

</td></tr>
</table>

Data Analysis

To report teachers' attention to and flexibility with referent units, we computed the percentage of responses that fell in each of the three groups based on teachers' explanations. To investigate the relationship between teachers' attention to and flexibility with referent units, we created a contingency table and used a chi-square test to examine the relationship between them. Finally, to examine the relationships among teachers' overall knowledge of fractions, their attention to and flexibility with referent units, and the professional background variables, we conducted a linear regression in which the total score on the fraction measure was predicted by teachers' attention to and flexibility with referent units and the aforementioned background variables.

Results

Teachers' Attention to Referent Units

As shown in Fig. 4, 53.7% of the teachers explicitly attended to referent units in their responses and reported that 1/3 could be greater than 1/2 if these fractions referred to different wholes or units (see Group 3 for Attention to Referent Units in Fig. 4). For instance, one teacher explained, "It is. If I am comparing two different-sized objects, then 1/3 may be greater than 1/2. For example, 1/3 of a large avocado may be larger than 1/2 of a small avocado."

On the other hand, almost one-fifth of the sample (19.5%) was classified as Group 2 because they used the same-sized referent unit to justify the impossibility of 1/3 being greater than 1/2. For example, one teacher wrote,

> No, it is not possible for one third to be greater than one half. One third is a smaller fractional piece. For example, if I have two medium-sized pizzas, if I split one in half and the other into thirds, the slices of the pizza split into thirds will be smaller than the slices of the pizza split into halves.

Finally, 26.8% of the sample ($N = 66$) provided responses classified as Group 1. These teachers either did not provide any justification ($N = 17$) or did not use an explicit referent unit ($N = 49$). The two most common approaches among those who provided an explanation without explicitly noticing the referent wholes were to find a common denominator for both fractions ($N = 18$) or to convert these fractions to decimals or percentages ($N = 13$). For example, one teacher explained, "No, it is not possible for one third to be greater than one half. To easily compare these fractions, you can find

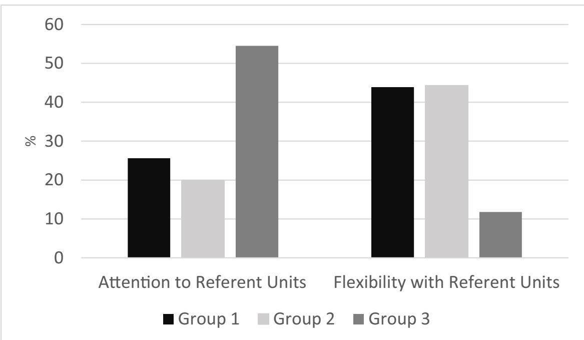


Fig. 4 Percentages of teachers demonstrating attention to and flexibility with referent units

common denominators, 2/6 and 3/6. The one half will always be greater than the one third."

Teachers' Flexibility with Referent Units

We found that only 29 teachers (11.8% of the sample) coordinated different units and reported that the referent unit for the multiplier (i.e., 1/3) was the multiplicand (i.e., 1/4 of the rectangle, not the entire rectangle; see Group 3 for Flexibility with Referent Units in Fig. 4). They also pointed out that the referent unit for the product (i.e., 1/12) was the whole rectangle. For example, one teacher explained,

> Next, Ms. Bender should divide the shaded rectangle vertically into 3 equal-sized pieces, and shade in more darkly (or identify by some other means) one of the 3 new partitions within the shaded region, explaining that she has now taken 1/3 of 1/4. She should ask the students to count how many total partitions there are within the rectangle now that it has been divided twice, and then she should challenge students to quantify what the single darkly shaded region represents numerically compared to the total partitions (1/12).

The responses of 109 teachers (44.3% of the sample) were categorized as Group 2 because they used the overlapping method. This meant they used the same referent whole (i.e., the whole rectangle) for all the fractions in the fraction multiplication situation. For instance, one teacher in Group 2 wrote,

> The next step I would take would be to ask students how we could also show 1/3 on this drawing. I would be looking for someone to say we can draw three even vertical lines on the drawing to represent thirds and shade one in to represent 1/3. We would then have a discussion about how now if we count our sections, we now have 12 total boxes and only 1 of those boxes was shaded in from both the 1/4 picture and the 1/3 picture.

Note that teachers using this strategy considered the referent unit to be the same (the whole rectangle) for all the fractions involved. Thus, even though this strategy correctly identified the referent unit for the product (1/12), it simply mimicked the procedure of the fraction multiplication algorithm rather than focusing on the conceptual underpinnings of the operation (i.e., 1/3 of 1/4).

Table 3 Number of teachers in the explanation categories for attention to and flexibility with referent units

Attention to referent units	Flexibility with referent units		
	Group 1	Group 2	Group 3
Group 1	38	23	5
Group 2	24	20	4
Group 3	46	66	20

Finally, the responses of 108 teachers (43.9% of the sample) were categorized as Group 1 because they either indicated not knowing how to model fraction multiplication or that their explanation was not clear about what referent whole they used for the multiplier and the product. As an example of the lack of clarity regarding what referent unit was used, one teacher wrote, "Next, Ms. Bender would have to split the fourths into thirds either long ways or sideways so that there are 12 equal pieces in the square without adding on any."

Relationship Between Attention to and Flexibility with Referent Units

So far, we have reported on teachers' attention to and flexibility with referent units separately. Looking across individual teachers' responses to both tasks, we found a statistically significant relationship between the two aspects of teachers' understanding of referent units, $\chi^2(4) = 10.8$, $p = .03$. As shown in Table 3, 38 teachers (15.5% of the sample) provided responses categorized as Group 1 for both tasks, indicating that they demonstrated neither attention to nor flexibility with the referent units. In contrast, only 20 teachers (8.1% of the sample) showed both attention to and flexibility with the referent units.

Linking Teachers' Attention to and Flexibility with Referent Units to Their Performance on the Fraction Measure

As mentioned, we also captured teachers' overall understanding of fraction concepts by using a set of tasks that were not specifically designed to capture their understanding of referent units, to investigate the extent to which teachers' understanding of referent units was associated with teachers' overall knowledge of fractions. As shown in Table 4, teachers' attention to and flexibility with referent units were linked to their performance on the fraction measure. Specifically, teachers whose responses included

Table 4 Linear regression results for teachers' understanding of referent units predicting teachers' knowledge of fractions

Predictor	Teachers' score on the fraction measure
Attention to RU (Group 2)	**0.33*** (.09)**
Attention to RU (Group 3)	**0.28*** (.07)**
Flexibility with RU (Group 2)	**0.20** (.06)**
Flexibility with RU (Group 3)	**0.20* (.10)**
Multiple-subject credentials	0.002 (.09)
Math teaching credential	0.16 (.11)
Middle school math teacher	0.13 (.09)

Note. $N = 238$ for all models. The numbers in parentheses are standards errors. *RU* referent units. Boldface indicates statistically significant results

*$p < 0.05$, **$p < 0.01$, ***$p < 0.001$

specific referent units (i.e., Groups 2 and 3 for the attention to referent units task) achieved a statistically higher score on the fraction measure than did those who did not provide a referent unit in their answers (Group 1 for the attention to referent units task). However, the differences in the performance of teachers whose responses were categorized in Groups 2 and 3 for attention to referent units were not statistically significant. This may indicate that teachers whose responses included a specific referent unit, regardless of whether they agreed 1/3 could be greater than 1/2, performed similarly on the fraction measure.

Teachers who used the overlapping method or who coordinated different referent units to model fraction multiplication (i.e., Groups 2 and 3 for the flexibility with referent units task) performed statistically better than did those who did not show flexibility with referent units (i.e., Group 1). As in the case of attending to the referent unit, teachers who used the overlapping method (Group 2) or who coordinated different units in fraction multiplication modeling (Group 3) did not perform statistically differently from one another.

Discussion

In the present study, we examined U.S. in-service teachers' attention to and flexibility with referent units. We particularly focused on how teachers' attention to and flexibility with referent units were linked as well as their relationship to teachers' performance on a separate measure capturing their overall understanding of fractions. Our results suggest that attention to and flexibility with referent units are important and yet distinct elements. This result is in line with the theoretical justification of von Glasersfeld (1981) that noticing plays a significant role in understanding referent units. Our study confirms that teachers who attended to the referent units in a situation where unit wholes for fractions were not given performed statistically better on the fraction measure than did those who did not state a referent unit in their answers.

We believe this is an important finding with implications for teacher education. Recall that prior research has shown some teachers do not seem to use the same size whole for fraction comparisons (Bartell et al., 2013). Thus, we argue that teachers need more targeted learning opportunities to increase their attention to referent units. Furthermore, more professional learning opportunities are needed to equip teachers with the necessary skill to incorporate strategies that will help their students attend to referent units.

Our findings regarding teachers' flexibility with referent units are in line with prior work in that only a small percentage of teachers coordinated different referent units (e.g., Baek et al., 2017; Lee, 2017; Lee et al., 2011; Luo et al., 2011; Son & Lee, 2016; Webel et al., 2016; Webel & DeLeeuw, 2016). Furthermore, as in past research (e.g., Izsák, 2008; Lee et al., 2011; Luo et al., 2011; Webel et al., 2016; Webel & DeLeeuw, 2016), teachers commonly used the overlapping method to model fraction multiplication. Our findings differ from prior work in that we provide evidence of the role of teachers' flexibility with referent units in their overall understanding of fractions. In particular, we found that teachers who used the overlapping method and those who flexibly coordinated different referent units performed similarly on the separate fraction measure. We believe the similar performance of these two groups of teachers requires further investigation. It is possible that teachers' prior learning opportunities are associated with how they modeled fraction multiplication. Unfortunately, the answer to this question is beyond the scope of our study. Thus, further research is needed to identify how teachers' flexibility with referent units is associated with their learning opportunities as well as the learning environment they create for their students.

We also illustrate the importance of these two aspects of referent units by providing evidence that those who demonstrated attention to and flexibility with referent units outperformed those who did not. Prior work has suggested that teachers' understanding of referent units seems to explain some of the differences in groups of teachers who show different levels of understanding of fraction arithmetic (Izsák et al., 2010). Our finding provides further evidence that both aspects of referent units are important indicators of teachers' understanding of fractions. Yet further research is needed to investigate how different aspects of teachers' understanding of referent wholes are associated with the learning environment teachers create for their students to understand fractions.

In conclusion, scholars agree on the importance of teachers' understanding of referent units. Yet this study contributes to the current literature by underscoring the importance of attending to referent units along with showing flexibility in coordinating different referent units. These findings also confirm our initial expectation that providing a drawn model or a word problem would capture a different aspect of teachers' understanding of referent units, given that the unit is provided to teachers in these conditions. We urge researchers to further investigate how different models, such as area and length, could affect teachers' flexibility with referent units as well as how teachers' attention to referent units could be captured in different situations. We hope that this study will draw attention to the need to create more learning opportunities in teacher education and professional development programs that will further develop teachers' attention to and flexibility with referent units.

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