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ORIGINAL ARTICLE

Preservice teachers' understandings of division and ratios in forming proportional relationships

İbrahim Burak Ölmez[1]

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Abstract
This study aimed at investigating how preservice teachers' understandings of division and reasoning about ratios support and constrain their formation of proportional relationships in terms of quantities. Six preservice teachers from a middle-grade preparation program in the USA were selected purposefully based on their mathematics performance in a previous course. An explanatory case study with multiple cases was used to make comparisons within and across cases. Two semi-structured interviews were conducted with each pair. The results revealed that preservice teachers who did not explicitly identify different meanings for division struggled to differentiate between the two perspectives on ratios. The results also showed that those teachers had difficulty forming proportional relationships while solving the proportion tasks. These results suggest that explicit identification of the meanings for both types of division is critical to keeping the two perspectives on ratios separate, which is a key aspect for a robust understanding of proportional relationships.

Keywords Division · Preservice teachers · Ratios and proportional relationships

Introduction

Ratios and proportional relationships are core topics of primary and secondary mathematics (e.g., Kilpatrick et al., 2001; National Council of Teachers of Mathematics (NCTM), 2000) and necessary for understanding diverse topics including linear functions, slope, and probability (e.g., Lobato & Ellis, 2010). Despite the importance of ratios and proportional relationships, past research has shown that both students and teachers struggle with these topics, and much of this research has investigated students' understanding of proportional relationships (e.g., Behr et al., 1992; Lamon, 2007). For example, based on the results of the National Assessment of Educational

✉ İbrahim Burak Ölmez
i.burakolmez@hotmail.com

1 Department of Mathematics and Science Education, MEF University, Istanbul, Turkey

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Progress (NAEP) (National Center for Education Statistics, 2013), more than half of the eighth-grade students in the USA responded to problems involving proportional relationships incorrectly. Past research has also shown that teachers' difficulties are often similar to those of students (e.g., Lamon, 2007). Specifically, teachers can perform poorly on tasks involving proportional relationships other than missing-value problems (e.g., Pitta-Pantazi & Christou, 2011; Riley, 2010); have difficulty coordinating two quantities in a proportional relationship (e.g., Orrill & Brown, 2012) and identifying directly proportional, inversely proportional, and nonproportional situations (e.g., Arıcan, 2019; Izsák & Jacobson, 2017); focus more on additive relationships than on multiplicative relationships between quantities (Ölmez, 2016); rely heavily on problem-solving strategies such as cross-multiplication as a rote computation approach (e.g., Arıcan et al., 2018); and guess at arithmetic operations and search for key words (e.g., Harel & Behr, 1995).

As opposed to rote computation strategies such as cross-multiplication when solving problems, proportional relationships can be conceived of from two perspectives based on the multiplicative nature between quantities: the variable-parts perspective and multiple-batches perspective (Beckmann & Izsák, 2015). The variable-parts perspective indicates a fixed number of parts that can vary in size. For example, in a fruit punch made of 3 cups of peach juice and 5 cups of apple juice, the relationship of peach juice to apple juice can be conceived as 3 parts to 5 parts. As long as the number of parts is fixed (3 parts to 5 parts), changing the size of each part (e.g., from cups to tablespoons) does not change the taste of the fruit punch. On the other hand, the multiple-batches perspective indicates varying the number of groups (or batches) that are made up of a pair of fixed amounts. For example, in a fruit punch made of 3 cups of peach juice and 5 cups of apple juice, a pair fixed amounts can be viewed as forming one batch consisting of 3 cups of peach juice and 5 cups of apple juice. Then, this fruit punch would taste the same if it was doubled (6 cups of peach juice and 10 cups of apple juice), tripled (9 cups of peach juice and 15 cups of apple juice), and so on. Weiland et al. (2021) also report that these two perspectives are two of the knowledge resources necessary for teachers' robust understanding of proportional relationships. Both perspectives will be described in detail later.

In the USA, the Common Core State Standards for Mathematics (CCSS; Common Core State Standards Initiative, 2010) recommends making sense of quantities and their relationships by using visual representations such as strip diagrams and double number lines. The CSSS recommends all students use double number lines and strip diagrams when solving tasks that involve proportional relationships. In this way, those representations can support the CCSS mathematical practices including making sense of problems, reasoning quantitatively and abstractly, and constructing viable arguments. In contrast to the CCSS recommendation, existing literature has revealed that teachers may use visual representations only to picture a final result, not to support a complete solution to a problem (e.g., Izsák, 2008).

In addition to ratios and proportional relationships, division is another core topic in primary mathematics and necessary for secondary mathematics. By viewing division as multiplication with an unknown factor, division has two primary types of meanings: partitive division and quotitive division (e.g., Greer, 1992). Even though the operation $v \div w$ is easily identified as division, its interpretation alters based on whether we are looking for the number of groups or for the size of each group. If we use the operation $v \div w$ to get the number of groups when v items are divided by w items in each group, then we are using *quotitive (how many groups) division*. If we regard the operation $v \div w$ to get the number of items in each group when v items are shared by w groups equally, then we are using *partitive (how many in each group) division* (Beckmann & Izsák, 2015). Interpreting division with multiple meanings is highly recommended in recent curriculum standards (e.g., CSSS, 2010).

Considering that proportional relationships are part of the *multiplicative conceptual field* (Vergnaud, 1983, 1988) — a web of interrelated topics including whole number multiplication and division, fractions, ratios, and more — I conducted a study to identify how preservice teachers' (PSTs) understandings of division and reasoning about ratios might support and constrain their formation of proportional relationships. In particular, I report results from two semi-structured interviews during which three pairs of PSTs worked on tasks involving ratios and proportional relationships. The following research question was addressed in this study:

- How do PSTs' understandings of division and reasoning about ratios support and constrain their formation of proportional relationships?

Significance of the study

Because partitive division involves an understanding of unit rate, which is based on proportional relationships (Jansen & Hohensee, 2016; Hohensee & Jansen, 2017) and both division and proportional relationships are located in the multiplicative conceptual field, I hypothesize that understanding of division might influence the ways that PSTs reason about proportional relationships. It is highly possible that PSTs' difficulties with proportional relationships have roots deep in the multiplicative conceptual field.

The current study makes two contributions. First, past research has not examined whether PSTs' understandings of division might influence their formation of proportional relationships. Although teachers often have been reported to have weak conceptual meanings for division (e.g., Ball, 1990; Simon, 1993; Timmerman, 2014), past research has not explored ways that teachers' facility with division might support and constrain their reasoning about proportional relationships. Second, a main finding reported across several studies is that teachers have persistent difficulties in determining whether two quantities are in a proportional relationship, especially in missing-value problems (e.g., Fisher, 1988; Izsák & Jacobson, 2017; Lim, 2009). Past research, however, has not reported how PSTs' reasoning about the two perspectives on ratios might support and constrain their ability to form proportional relationships. Therefore, the present study contributes to the literature by exploring how PSTs' understandings of division and reasoning about ratios might support and constrain their formation of proportional relationships.

Literature review

Research on students' understanding of division has acknowledged that students primarily use partitive division and only learn to use quotitive division later through instruction (e.g., Fischbein et al., 1985; Kaput, 1986; Lo & Watanabe, 1997). For instance, in one study by Fischbein et al. (1985), 5th-, 7th-, and 9th-grade students were given word problems involving multiplication and division and asked their opinion about the selection of the operation they would use. A main finding of this study was that students have two intuitive models for division: partitive and quotitive divisions, and the latter one is learned through instruction. In another study, Kaput (1986) asked students from grades 4 to 13 to generate division problems, and found that while 81% of the problems reflected partitive division, only 17% of them were quotitive division. In addition, no significant changes were found in the use of partitive division with the age of the students.

Similar to research on students' understanding of division (e.g., Kaput, 1986; Lo & Watanabe, 1997), research on teachers' understanding of division has demonstrated that teachers and PSTs did not have robust conceptual meanings for division including partitive and quotitive meanings, and they tended to think more of the partitive meaning of division than of the quotitive meaning (Ball, 1990; Ball et al., 2001; Graeber et al., 1986; Piel & Green, 2010; Rizvi & Lawson, 2007; Simon, 1993; Timmerman, 2014; Tirosh & Graeber, 1990). For example, in one study on 19 PSTs' understandings of division in primary and secondary grades programs, Ball (1990) conducted individual interviews by asking the PSTs to generate a word problem for 1 3/4 ÷ 1/2. Although 17 of the 19 teachers could perform the division correctly, only 5 of them were able to make sense of the situation by thinking in terms of quotitive division: how many halves there are in one-and-three-fourths. This indicates that most PSTs struggled with meanings for division with fractions, despite their ability to calculate the division operation correctly. Ball (1990) argued that one reason for the PSTs' difficulty might be their thinking only in terms of division, not quotitive division.

Similarly, in another study, Simon (1993) asked 33 PSTs in a primary grades program to write different word problems for division of 51 by 4 for which answers of 12 3/4, 13, and 12 were appropriate. While 74% of the problems were based on partitive meaning of division, 17% of those reflected quotitive meaning of division, implying the PSTs thought primarily in terms of partitive division. Another main finding reported by Simon (1993) was that PSTs were unable to make shifts in thinking between partitive and quotitive division. Moreover, Simon found evidence of numeric division, which is based only on obtaining the numeric answer, among the responses of PSTs, indicating a lack of understanding the meanings of division in terms of quantities.

In a recent study, Piel and Green (2010) asked the following problem of PSTs: "$8 \div 4 = x$. If the numbers 8 and 4 both represent cookies, solve the equation" (p. 73). In response to this problem, most of the PSTs (86%) incorrectly drew 2 cookies without showing how 8 cookies and 4 cookies were used in the Eq. $8 \div 4 = x$. Instead of drawing groups of 4 cookies exhausting a total of 8 cookies indicating quotitive division, those PSTs' responses focused on the final result, which is 2. This indicates PSTs' reliance on procedural understanding of the division operation (obtaining the numeric answer) instead of focusing on the meaning of division. Similarly, in another study by Jong and Magruder (2014) on 55 PSTs' understandings of multiplication and division, most PSTs struggled to generate word problems that involve quotitive division. About 30% of the PSTs either confused the two meanings for division or wrote incorrect word problems for $24 \div 8 = 3$ versus $24 \div 3 = 8$. It can be inferred from the results of these studies that PSTs have trouble understanding meanings for division that might lead to difficulties with ratios and proportional relationships.

Theoretical framework

The theoretical framework of this study is based on Beckmann and Izsák's (2015) mathematical analysis combining multiplication, division, and two perspectives on proportional relationships. Beckmann and Izsák considered the equation "$M \bullet N = P$" to be the number of groups times the number of units in each whole group equals the number of units in M groups (Fig. 1). Similar to the previous descriptions, looking for "how many groups" in the equation is called *quotitive division*, and getting "how many in each group" is known as *partitive division*.

M • N = P

(# of groups) • (# of units in each/one whole group) = (# of units in M groups)

Equation A $M \bullet N = x$ How many units in M groups of N? multiplication	**Equation B** $M \bullet x = P$ How many units in each/one group? division (partitive division)	**Equation C** $x \bullet N = P$ How many groups? division (measurement division)
Equation D $x \bullet y = P$ Inversely proportional relationship	**Equation E** $x \bullet N = y$ Variable number of fixed quantities proportional relationship (multiple batches)	**Equation F** $M \bullet x = y$ Fixed numbers of variable parts proportional relationship (variable parts)

Fig. 1 Mathematical analysis of ratios and proportional relationships (From Beckmann & Izsák, 2015, p. 19)

Beckmann and Izsák extended previous literature by defining two perspectives on ratios which connect the two meanings of division. First, according to the multiple-batches perspective, A units of the first quantity and B units of the second quantity can be viewed as composed units (Lobato & Ellis, 2010) or "batches" and the ratio A to B can consist of two quantities in which their amount is multiples of those fixed measurements (batches). Then, by regarding N as the value of $A \div B$, N is derived from the equation $x \bullet N = y$ that is *partitive (how many in each group) division*. Second, according to the variable-parts perspective, one can fix the number of parts for each of the two quantities with the condition that the number of parts stays the same, but the size of each part can vary. Then, by regarding M as the result of the operation $A \div B$, M is taken from the equation $M \bullet x = y$ that indicates *quotitive (how many groups) division*.

While the multiple-batches perspective is well studied in mathematics education literature (Lobato & Ellis, 2010), the variable part perspective has been largely overlooked despite its use in some Asian countries such as Singapore (Beckmann & Kulow, 2018). The different roles played by M and N in the form of $M \bullet N = P$ leads to several different ways of solving problems that involve proportional relationships (see Beckmann et al., 2015 for details).

I illustrate the two perspectives on ratios with the following Oil task:

> A fragrant oil was made by mixing 3 mL of lavender oil with 2 mL of rose oil. What other amounts of lavender oil and rose oil can be mixed to make a mixture that has exactly the same fragrance?

From the multiple-batches perspective, we can view 3 mL of lavender oil and 2 mL of rose oil as forming one batch in the double number line (Fig. 2), and then covariation between the two quantities as multiples of the original batch. As an example, 6 mL of lavender oil and 4 mL of rose oil would be two batches, and 1 mL of lavender oil and 2/3 mL of rose oil would be one third of a batch, and so on. Because the 3-to-2 ratio is preserved in forming those batches, each mixture would smell the same. In the multiple-batches perspective, the number of batches varies, but the size of each batch is fixed. To see how the multiple-batches perspective connects the two meanings of division, consider the following scenario asking how much rose oil should be mixed by 6 mL of lavender oil? In this perspective, we can either use *partitive (how many in each group) division* by dividing 3 mL by 2 mL to determine that 3/2 mL is needed for each batch. Or, we can use *quotitive (how many groups) division* by dividing 6 mL by 3 mL to determine that 2 batches are needed for the mixture.

From the variable-parts perspective, we can view the amount of lavender oil as consisting of 3 parts and the amount of rose oil as consisting of 2 parts in the strip diagram (Fig. 2), where all parts are the same size. We can view the covariation as varying the sizes of all parts simultaneously. As an example, by considering each part as 2 mL, the amount of lavender oil and rose oil would be 6 mL and 4 mL, respectively. Or, if we conceive of each part as 1/3 mL, then there would be 1 mL of lavender oil and 2/3 mL of rose oil. Because the 3-to-2 ratio is represented with 3 parts of lavender oil and 2 parts of rose oil, each mixture would smell the same. In this perspective, the number of parts is fixed, but the size of each part varies. To see how the variable-parts perspective connects the two meanings of division, consider

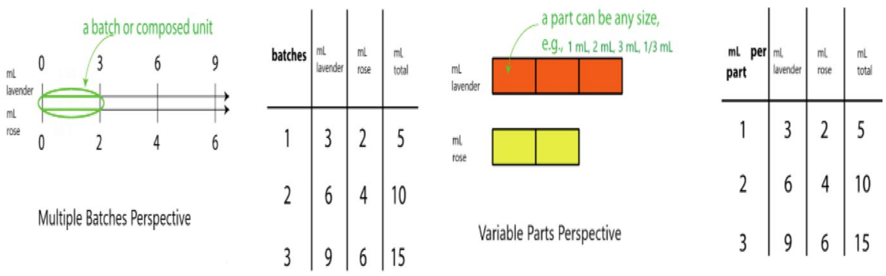

Fig. 2 Multiple-batches perspective with a double number line (on the left) and variable-parts perspective with a strip diagram (on the right) showing 3 to 2 ratio

the same scenario asking how much rose oil should be mixed by 6 mL of lavender oil? In this perspective, we can either use *partitive (how many in each group) division* by dividing 6 mL by 3 parts to determine that each part contains 2 mL. Or, we can use *quotitive (how many groups) division* by dividing 3 parts by 2 parts to determine that 3/2 as much rose oil as lavender oil is needed for the mixture.

The design of the mathematics content courses from which PSTs were recruited and the design of the interviews were based on the mathematical analysis presented above. Specifically, the mathematics content courses targeted PSTs' reasoning about multiplication and division in terms of quantities and their reasoning from the two perspectives by using strip diagrams and double number lines.

Method

An *explanatory case study* (Yin, 1993) was used to explore PSTs' understandings of division, ratios, and proportional relationships, and multiple cases were selected to improve generalizability and external validity of the research (Gay et al., 2008). Case studies are appropriate when researchers want to examine a particular phenomenon with descriptive questions such as what happened? or explanatory questions such as how did something happen? (Mills & Gay, 2019). Because this study aimed at exploring PSTs' reasoning in-depth, each individual PST formed a case. This is an initial exploratory study that attempts to explain how PSTs' understandings of division and their ability to differentiate the two perspectives on ratios might support and constrain their formation of proportional relationships.

Participants and context

The present study was conducted with three pairs of PSTs from a middle-grades preparation program at one large university in the Southeast USA. The PSTs were preparing to teach Grades 4–8 (ages 10–14). The middle-grades teacher preparation program is an undergraduate, 4-semester program that includes coursework in two subject area emphases (from among mathematics, science, language arts, and social studies) and teaching methods related to middle grades' curriculum and students. The program requires only a standard first semester calculus course. Other than the first calculus course, the PSTs had to take a sequence of three paired content and method courses. At the time of the study in 2012, PSTs were already in their fourth year of college and had already completed two paired content and method courses, one on number and operations where the main focus was on multiplication, division, and fractions, and one on geometry. PSTs were enrolled in paired content and method courses in algebra when this study was conducted. At the beginning of the algebra course, six PSTs were recruited purposefully by the same instructor of the number and operation course and the algebra course on a voluntary basis. Each two PSTs were paired with similar performance in the previous course on number and operations (two PSTs with low performance, two PSTs with medium performance, and two PSTs with high performance). The PSTs had not received any instruction on ratios and proportional relationships before the algebra course. The PSTs were interviewed in pairs with similar performance because of two reasons. First, PSTs were already familiar with working in groups based on their classroom practices. By considering each other's points of view, it was possible to obtain richer information about the PSTs' range of ideas during the interviews. Second, being interviewed by the instructor of the algebra course could make the PSTs feel unsettled. Thus, it was thought that having a group partner with similar performance could make the PSTs feel better because their peers also had to think hard about the interview tasks.

The textbook for the middle-grades content course was *Mathematics for Elementary Teachers* (3rd ed.) (Beckmann, 2011), and the PSTs in the middle-grades preparation program were taught ratios and proportional relationships at the beginning of the algebra content course. The aim of this content course was to develop PSTs' understandings of multiplication, division, fractions, and ratios and proportional relationships in ways consistent with the CCSS (CCSS, 2010). Specifically, the content course gave PSTs problem situations with quantities and asked them to explain their solutions in group discussions, in homework, and in exams, rather than only solving problems. The course also addressed both the multiple-batches perspective with the use of double number lines, and the variable-parts perspective with the use of strip diagrams for representing quantities that involve proportional relationships illustrated in Fig. 2.

Data collection

The instructor of all three content courses conducted two semi-structured (e.g., Bernard, 1994, chapter 10) 1-hour interviews with each pair of PSTs who were paid $25 for each hour as an incentive. The first interviews were conducted after 3 weeks of instruction where the two perspectives on ratios were introduced. The second interviews took place nearly 3 weeks before the final exam when all topics of the course were covered. The goal for the first interviews was to explore the PSTs' understandings of the two perspectives on ratios individually and in contrast to each other. In particular, I concentrated on each PST's understanding of the meanings for division. The goal for the second interviews was to explore how the PSTs formed proportional relationships. The analysis of each pair proceeds chronologically to obtain evidence for their use of division and the perspectives on ratios that they used.

Table 1 presents the tasks used in this study. For the tasks in the first interview, I expected the PSTs to reason from each perspective on ratios successfully without using any words or visual representations that were part of the other perspective. At the same time, I expected them to explicitly identify different meanings for division when solving the tasks. For the task in the second interview, I expected the PSTs to determine how the 7-to-5 ratio is preserved when some amount of one quantity is added to the mixture.

In each interview, a separate piece of paper was given to each PST for each task. The interviewer read the task, and PSTs worked together and explained their reasoning out loud. Each interview was video-recorded using two cameras, one for capturing the interviewer and the pair and one for capturing the written work of the pair. Then, two video files from two cameras were combined into one video file for restored view (Hall, 2000) and transcribed verbatim. Also, the written work of the pairs was collected. Hence, the data in this study consist of two video-recordings of interviews for each pair, transcriptions of the interviews, and the written work of the PSTs.

Data analysis

I used a thematic analysis approach (Boyatzis, 1998) to analyze the interview data. After the data were collected, I took multiple passes through the data by reviewing the transcripts side-by-side with the videos. I concentrated on the PSTs' words, gestures, and inscriptions to gather evidence about their thinking processes. To analyze the transcripts, I wrote detailed summaries of each video, and attempted to identify the mathematical ideas in the PSTs' thinking. After describing how each PST reasoned about each task using open coding, I made

Table 1 Interview tasks

Interview 1
Task 1: A fragrant oil was made by mixing 3 mL of lavender oil with 2 mL of rose oil. What other amounts of lavender oil and rose oil can be mixed to make a mixture that has exactly the same fragrance?
Task 2: What does it mean to say that lavender oil and rose oil are mixed in a 3-to-2 ratio?
Task 3: If I give you some amounts of lavender oil and rose oil, how can you tell if they are mixed in a 3-to-2 ratio? For example, consider each of these mixtures:
12 mL of lavender oil, 8 mL of rose oil; 21 mL of lavender oil, 12 mL of rose oil;
14 mL of lavender oil, 8 mL of rose oil; 5 mL of lavender oil, 3 mL of rose oil
Interview 2
Scenario: To make 14-karat gold jewelry, people mix pure gold with another metal such as copper to make a mixture that we will call "jewelry-gold." Jewelry-gold is made by mixing pure gold with copper in the 7-to-5 ratio
Task 4: There was some jewelry-gold that was made by mixing pure gold and copper in a 7-to-5 ratio. Then another 4 grams of pure gold was added to the jewelry-gold mixture. What can you say about this new mixture?

a list labeling the ideas, concepts, and ways of reasoning each PST used as he or she worked on the task. Then, I compared the lists across multiple PSTs to make connections and finalize the set of themes.

In the first pass, I noticed that the PSTs struggled to reason from the two perspectives on ratios. Specifically, there were cases in which the PSTs were focusing on the variable-parts perspective by drawing strip diagrams, but their reasoning about quantities and use of language was related to the multiple-batches perspective. Mixing the two perspectives in such a way instead of differentiating them successfully indicated some weaknesses in the PSTs' reasoning about proportional relationships. At the same time, I realized that those PSTs also had a hard time explicitly identifying different meanings for division when solving the proportion tasks. As I took more passes through the data, it became increasingly apparent that the PSTs' reasoning in the first interviews was substantially different in terms of the formation of proportional relationships than their reasoning in the second interviews.

In terms of reasoning about ratios from the two perspectives, substantial diversity occurred in two situations: (a) the PSTs could *keep the two perspectives separate* by differentiating them successfully using appropriate reasoning and wording with each perspective or (b) the PSTs *mixed the two perspectives* with inappropriate reasoning and wording. Specifically, reasoning about the fixed number of sizes and varying number of groups or batches and using appropriate wording about replication or iteration of the batches reflect the multiple-batches perspective. On the other hand, interpreting the fixed number of parts with each part varying in size and using appropriate wording about changing the size of each part reflect the variable-parts perspective (Ölmez, 2016). For instance, when responding to the Oil tasks in the first interview, one PST could consider each part of the strip diagram with 1 mL to preserve the 3-to-2 ratio between lavender oil and rose oil. If he or she interprets covariation of the relationship between lavender oil and rose oil in terms of replication of parts of a strip diagram as if they are batches, instead of varying the size in each part by keeping the number of parts fixed, this indicates his or her *mixing the two perspectives* and not differentiating the two perspectives successfully (Fig. 3). To keep the two perspectives separate, this PST would either need to keep the number of parts in the strip diagram by changing the amount in each part (variable-parts perspective) or draw a double number line and iterate the fixed number of sizes (3 mL of lavender oil and 2 mL of rose oil) by changing the number of batches (multiple-batches perspective; see Fig. 2).

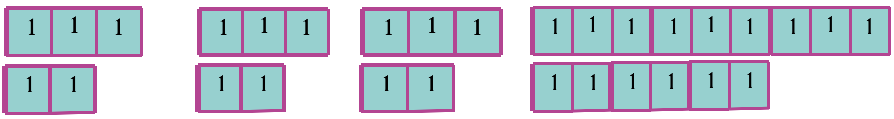

Fig. 3 Mixing the two perspectives on ratios for the Oil tasks

Results

In this section, I present a summary of all three pairs of PSTs' understandings of division and their reasoning about ratios from the two perspectives in the first interviews, and their formation of proportional relationships in the second interviews. In particular, I demonstrate that PSTs' explicit identification of different meanings for division co-occurred with their ability to differentiate the two perspectives on ratios successfully in the first interviews, and those PSTs could form proportional relationships in the second interviews. On the other hand, I also demonstrate that PSTs who did not identify different meanings for division had difficulties keeping the two perspectives separate and struggled to form proportional relationships.

I illustrate three pairs of PSTs ranging from less to more proficient in their understandings of division and reasoning from the two perspectives on ratios. The least proficient pair was Pair 1, Lisa and Tess, who were not clear about different meanings for division, and could not differentiate the two perspectives successfully in the first interview. At the same time, they had a hard time when determining whether addition of 4 grams of gold to the initial mixture would change the 7-to-5 ratio in the second interview. Pair 2, Chip and Amber, performed better than Pair 1 in terms of understandings of division and reasoning from the two perspectives in the first interview. In the second interview, they were able to determine that addition of 4 grams to the initial mixture would change the 7-to-5 ratio with prompting from the interviewer. The most proficient pair was Pair 3, Amy and Paul, who could explicitly identify both meanings for division and keep the two perspectives separate in the first interview. They could also determine that adding 4 grams to the initial mixture would change the 7-to-5 ratio in the second interview.

Lisa and Tess

Summary for Lisa and Tess

Lisa primarily used partitive division, and Tess only used quotitive division. While Lisa did not explicitly identify different meanings for division during the first interview, she also could not keep the two perspectives separate on ratios when bringing her multiple-batches perspective thinking into reasoning about strip diagram. Within this pair, Tess performed better than Lisa regarding the use of division and reasoning from the two perspectives in the first interview. In the second interview, both Lisa and Tess struggled to determine whether adding 4 grams of gold to the initial mixture would change the 7-to-5 ratio, and they received prompting from the interviewer to find the result of 20/7 grams of copper for keeping the same 7-to-5 ratio.

Interview 1 for Lisa and Tess

In Task 1, both Lisa and Tess reasoned from the multiple-batches perspective by drawing a ratio table and by making iterations of the original batch as follows (Fig. 4):

> Lisa: For every 3 mL of lavender oil in the mixture we have 2 mL of rose oil and those are the values that make up one batch of the mixture. So if we want to double the initial batch, then we just multiply each value of the initial ratio by the number of batches… So, it's kind of like additional copies.

> Tess: Yeah, whatever your number of batches, you're just multiplying that by your first batch, so your original ratio to get your other ratios that are the same.

When the interviewer asked how they knew that it made the same fragrance, Lisa stated that "If you take each ratio and then divide it by the number of batches, then it will give you the initial ratio," indicating an evidence for partitive division. As a follow-up question, the interviewer asked whether there was another way to explain why the two mixtures would smell the same. Tess responded that "For every 3 like you have 3, 3, 3, 3, 2, 2, 2, 2 if you take those away, they're still the same as the first batch.," indicating an evidence for repeated subtraction as the form of quotitive division.

In response to Task 2 asking what a 3-to-2 ratio meant in the context of 3 mL of lavender oil and 2 mL of rose oil, while Lisa transitioned from the multiple-batches perspective to the variable-parts perspective by proposing a strip diagram, Tess continued reasoning from the multiple-batches perspective as follows:

> Tess: It means that for every 3 mL of lavender oil, you have 2 mL of rose oil so like because there the ratio is 3-to-2 you always have to make when you're making that second batch.

> Lisa: You can go the parts approach where for every 3 parts lavender oil you have 2 parts rose oil and then any volume quantity could represent.

A moment later, the interviewer asked whether the expression "for every 3 mL of lavender oil, there are 2 mL of rose oil" was related to their strip diagrams (Fig. 5). The following was an evidence that Lisa, but not Tess, struggled to keep the two perspectives separate:

> Tess: Here [points to her strip diagram on the right] you're looking at how many in each group, so you always have 3 groups and you always have 2 groups. We are just changing the amount in each group to add.

Batches (#)	1	2	3	4
mL lavender oil	3	6	9	12
mL rose oil	2	4	6	8

Fig. 4 Lisa's drawing of a ratio table

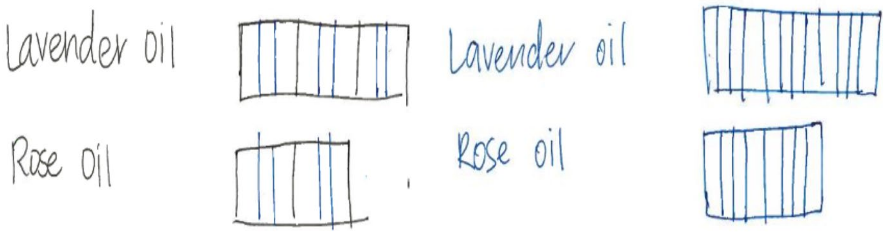

Fig. 5 Lisa's strip diagram (on the left) and Tess' strip diagram (on the right)

> Lisa: I guess here [points to her strip diagram on the left] if … you told me for every 3 parts there were 2 parts, I'm trying to think okay when I add 3 here then I'm going to add 2 here. Whereas, really you're increasing by the same amount.

Lisa's use of "for every" wording, when explaining her strip diagram, suggested iterations of multiples of parts by keeping the size of each part fixed but changing the number of parts, which was compatible with the multiple-batches perspective. Thus, her bringing multiple-batches thinking into reasoning about strip diagram indicates evidence for her difficulty of keeping the two perspectives separate. Moreover, Lisa said that she always thinks in terms of "repeated addition" of batches in these problems, which reflects the multiple-batches perspective as her primary way of reasoning about ratios.

In Task 3 that asks whether 12 mL of lavender oil and 8 mL of rose oil are in a 3-to-2 ratio, Tess demonstrated another evidence for her use of quotitive division as follows:

> Tess: So I guess you could if you divide by, your lavender by 3 and your rose by 2, if you get this same number, then it is going to be like the same mixture.

These data show that for Tess, 12 mL of lavender oil and 8 mL of rose oil were in a 3-to-2 ratio because they corresponded to the same batch (fourth batch), not different batches. So, Tess was attempting to find the number of batches, indicating quotitive division.

On the other hand, Lisa proposed dividing 12 mL of lavender oil and 8 mL of rose oil by 4 to attain a 3-to-2 ratio as follows:

> Lisa: I was just saying if we could divide both values by a constant to get 3 to 2. So I saw it like you can divide 12 and 8 both by 4 to get 3 to 2.

These data did not make Lisa's understanding of division clear in this task. However, her suggestion of dividing both 12 mL and 8 mL by 4, as opposed to Tess who proposed dividing 12 mL by 3 and 8 mL by 2, indicated that Lisa's use of division was either partitive or numeric division. Later, when the interviewer asked whether their use of division was similar because their arithmetic was different in their explanations (dividing by 3 and 2 for Tess versus dividing by 4 for Lisa), Lisa said that their reasoning for the use of division was similar. This indicates another evidence for Lisa's difficulty of explicitly identifying different meanings for division.

Interview 2 for Lisa and Tess

In response to Task 4, Lisa and Tess both explained incorrectly that 4 grams would need to be added to both gold and copper for keeping the ratio same. After a while, Tess thought that they would have to know the amount of the mixture in order to find how much copper should be added for 4 grams of gold to keep the same ratio, whereas Lisa was not sure how to determine the appropriate amount of copper that needed to be added (see Fig. 6 for Lisa's strip diagram). Later, they also thought that the problem could not be solved because 4 grams was a fixed amount, but parts of strip diagram could change.

Then, the interviewer helped Lisa and Tess notice that whatever the amount of the mixture is, addition of 4 grams of gold requires addition of 20/7 grams of copper to preserve the 7-to-5 ratio. With such an aim, the interviewer asked if each part was 17 grams, the ratio would still be 7-to-5. They agreed because "you still have 7 parts and 5 parts." The interviewer continued by asking what if 7 parts of gold equaled 4 grams, rather than adding 4 grams, then how much copper they would need. For this question, Lisa answered quickly as "4/7 times 5." Then, the interviewer went back to the original question asking how much copper needs to be added in the case of 7 grams of gold and 5 grams of copper with the addition of 4 grams of gold. Lisa and Tess found the correct answer of 20/7 grams of copper a second time, by subtracting 5 total grams of copper from 55/7 grams of initial copper. When the interviewer asked whether finding the answer of 20/7 grams of copper second time was a coincidence, Tess explained that "for every 4 grams, we add 20/7 copper," but whether Lisa understood that this relationship held regardless of the amount of the initial mixture was unclear.

Chip and Amber

Summary for Chip and Amber

While Amber primarily used partitive division, Chip used both partitive and quotitive divisions in the first interview. Despite Amber's difficulty of explicitly identifying different meanings for division, they performed better than Lisa and Tess in terms of understandings of division and reasoning from the two perspectives on ratios. Although Chip could explain clearly that "for every" language was not suitable for thinking about strip diagrams that involve the variable-parts perspective, Amber had difficulty of keeping the two perspectives separate. In the second interview, they were able to determine that addition of 4 grams to the mixture would change the ratio with prompting from the interviewer. While Chip

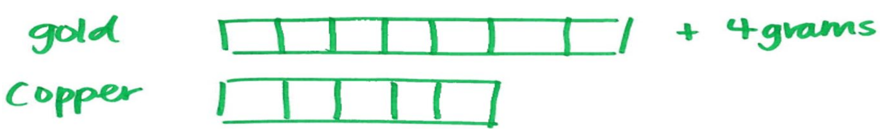

Fig. 6 Lisa's strip diagram in the second interview

was able to solve the task correctly, he could not see initially that for every 4 grams of gold, there must be 2.8 (or 20/7) grams of copper regardless of the amount of initial mixture. On the other hand, Amber had difficulty of making this connection.

Interview 1 for Chip and Amber

In Task 1, both Chip and Amber reasoned from the multiple-batches perspective by drawing a double number line (Fig. 7) and using whole number multiples of 3 and 2 to obtain larger batches:

> Chip: This is one batch… Say this is the sixth batch [points to 6 on the double number line], then all we're going to do is our initial ratio 2-to-3 times that 6… and you can keep going as high as you want to.

> Amber: … If you were only giving 2-to-3, and you want us to figure out the seventh batch, you would just multiply the number of batches by each number in the original ratio.

In addition to their reasoning from the multiple-batches perspective, Amber's dividing 12 mL and 18 mL by the number of batches in Fig. 7 indicated partitive division. When the interviewer asked them to explain the same fragrance without using the word ratio or proportion, Chip demonstrated evidence for his use of quotitive division as follows:

> Chip: So even if you have a huge tub of it, if you can separate it out to where 3 lavender goes to 2 rose and just keep separating it out, then eventually get to where there is none left. You don't have any leftovers but you just have a bunch of groups of the 2-to-3 like oil, then that would mean that it would smell the same.

These data show that Chip used quotitive division because he was looking for the number of groups when he divided the total amount by the amounts in each group.

An exchange later, when the interviewer asked whether there were other ways to show that lavender oil and rose oil had the same fragrance, Chip proposed the drawing of the following strip diagram that indicates his use of the variable-parts perspective and partitive division (Fig. 8):

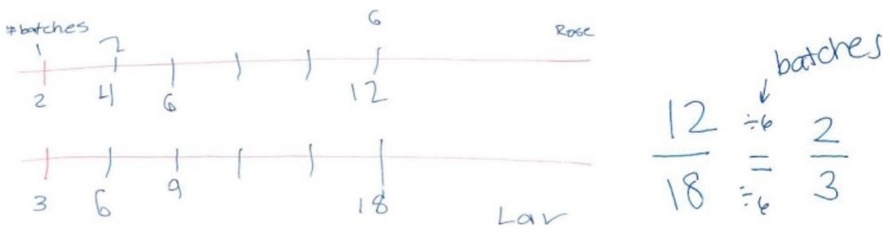

Fig. 7 Chip's double number line (on the left) and Amber's use of partitive division (on the right)

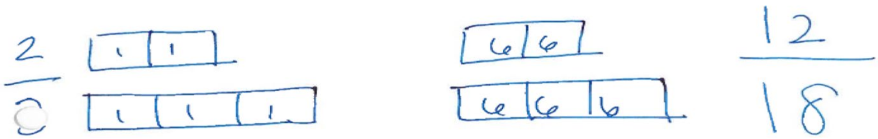

Fig. 8 Chip's proposed strip diagram drawing

> Chip: You could do it like 2 pieces, 3 pieces and then just say how much is in one piece. ... each has one cup in it. Now, in this next picture, we still have the same 2-to-3 ratio, we still have two parts of the rose and three parts of the lavender but now each part is worth 6 cups instead of one cup. So, 2/3 is equivalent to 12/18.

These data show that Chip used the variable-parts perspective by keeping the number of parts fixed and varying the size of each part; and, his use of "how much is in one piece?," in addition to dividing 2 mL by 2 parts and 3 mL by 3 parts, indicated partitive division.

In response to Task 2 asking what a 3-to-2 ratio meant in the context of 3 mL of lavender oil and 2 mL of rose oil, Chip consistently used "for every language" as follows:

> Chip: 3-to-2 ratio just means for every 3 lavender you have 2 rose. So if you have 50 things laying on the table, … if you can just pull 3 of the lavender and 2 of the rose and place them together … if you get to a point where there is none left and you have made like 10 groups of that 3-to-2.

An exchange later, when the interviewer asked whether having 51 cups total was possible, Chip made a strip diagram with 3 parts and 2 parts where each part is 10.2 cups. He then said that "Drawing there is not really what I was talking about. The way I'm talking about it would be like the individual batches." Thus, Chip seemed to know that his idea based on "for every" language was compatible with the multiple-batches perspective instead of the variable-parts perspective. This indicated an evidence for his keeping the two perspectives separate.

In Task 3 that asks whether 12 mL of lavender oil and 8 mL of rose oil are in a 3-to-2 ratio, Chip and Amber both divided 12 mL of lavender oil and 8 mL of rose oil by 4 to see whether the result would give a 3-to-2 ratio. They both said that they simplified the given numbers, which indicates evidence for numeric division (Fig. 9).

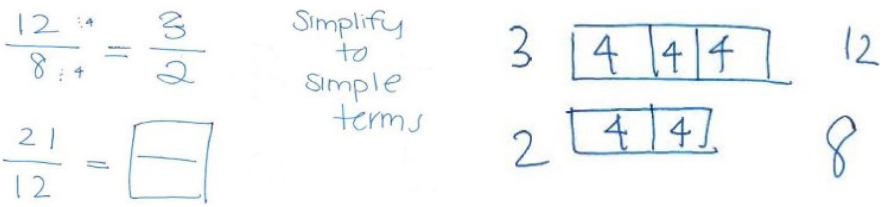

Fig. 9 Amber's use of numeric division (on the left) and Chip's proposed strip diagram (on the right)

A moment later, when the interviewer asked if there was another way to solve this task, Chip proposed drawing the strip diagram in Fig. 9, and Amber drew it based on his explanation. In addition to Fig. 9, Chip said that "You could redraw your picture of like the 12 to 8, like the 3 pieces and then 2 pieces and then just say each piece is worth 4." This demonstrates further evidence for his use of partitive division and reasoning from the variable-parts perspective. Amber, on the other hand, gave the following explanation using the multiple-batches perspective, despite her strip diagram in Fig. 9:

> Amber: You're just creating this [points to her strip diagram in Figure 9] 4 times… You're just adding the 3-to-2 four times to get to 12/8ths so (inaudible) all the same. You're just increasing the number of batches of the 3-to-2.

These data show that Amber used the multiple-batches thinking to reasoning about the strip diagram because she focused on making iterations of multiples of parts by keeping the size of each part fixed but changing the number of parts. She also considered 3 parts of lavender oil and 2 parts of rose oil in Fig. 9 as one batch, which was compatible with the multiple-batches perspective. Thus, Amber had trouble keeping the two perspectives separate.

Interview 2 for Chip and Amber

In response to Task 4, Chip and Amber both quickly noticed that the 7-to-5 ratio would change if 4 grams of gold were added to the mixture. When the interviewer asked whether the new ratio would be 11-to-5 if 4 grams of gold were added, Amber made a numeric calculation by doubling 11 and 5 and explained that the new ratio would not be 11-to-5. A moment later, the interviewer asked how much copper would need to be added to keep the 7-to-5 ratio in the mixture. Chip drew a strip diagram in Fig. 10 and responded correctly that 2.8 (or 20/7) grams of copper would be needed. First, he basically assumed each part as 1 gram and divided 11 grams of gold by 7 parts to obtain the number of grams in each part. Then, he multiplied the result, 11/7, by 5 parts to get the total amount of copper, subtracted this amount from 5 grams of copper, and found 2.8 (or 20/7) grams of copper as the answer. On the other hand, Amber did not draw a strip diagram, but only made numeric calculations.

A moment later, the interviewer asked what would happen if there were 10 grams in each part for Chip's strip diagram in Fig. 10 instead of 1 gram. Again, Chip did the same calculations and found the same answer of 2.8 (or 20/7) grams of copper. Then, he concluded that "So really what that says is for whatever number you have, if you add 4 grams of gold, then you're adding 2.8 grams of copper." This indicates that regardless of the

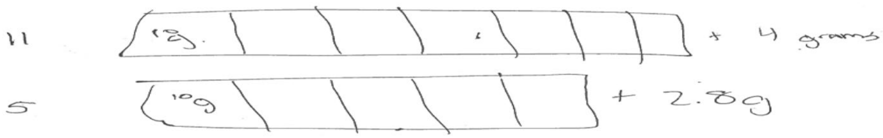

Fig. 10 Chip's strip diagram for keeping the 7-to-5 ratio same

initial amounts of gold and copper, Chip knew that 2.8 grams of copper need to be added for keeping the same 7-to-5 ratio. He also said that this understanding emerged during the interview. On the other hand, Amber had difficulty finding the same answer when there were 10 grams in each part. Because she was not sure whether for every 4 grams of gold, there needed to be 2.8 grams of copper, she checked her arithmetic by using a calculator. This indicates that Amber had greater difficulty solving Task 4 than did Chip.

Amy and Paul

Summary for Amy and Paul

While Amy used both partitive and quotitive divisions, Paul only used partitive division in the first interview. Amy clearly identified different meanings for division. They both reasoned from the two perspectives and could keep them separate on ratios. In terms of understandings of division and reasoning from the two perspectives on ratios, they were the most proficient pair. In the second interview, they were able to determine that addition of 4 grams to the mixture would change the ratio without any prompting from the interviewer. They could see that for every 4 grams of gold, there must be 20/7 grams of copper regardless of the amount in the initial mixture.

Interview 1 for Amy and Paul

In Task 1, Amy reasoned from the multiple-batches perspective by drawing the following ratio table:

> Amy: To make the same smell it can be any number and recipes or batches and multiply that number times 3 to find how much, how many mL of lavender oil multiply it by 2 to find out how many mL of rose oil.
>
> Amy: Because we kept these in the same ratio, like no matter which of these little sets of numbers you choose, they simplify to 3 and 2.

These data show that Amy generated whole-number multiples of 3 and 2 to obtain larger batches by keeping the size of each batch (or group) fixed. Her dividing the amount of oil by the number of batches in Fig. 11 also indicates an evidence for her use of partitive division. Similarly, Paul reasoned from the multiple-batches perspective by making multiples of 3 and 2 with the following double number line (Fig. 12):

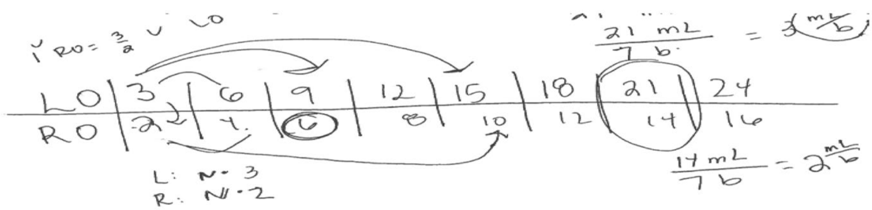

Fig. 11 Amy's drawing of a ratio table

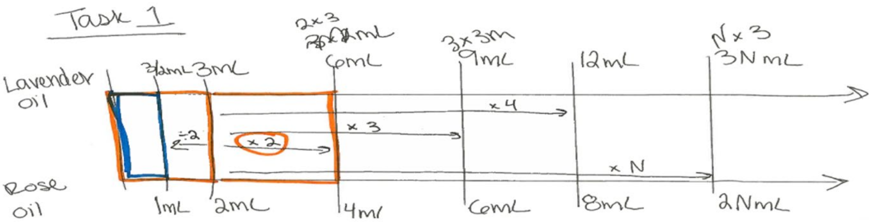

Fig. 12 Paul's drawing of a double number line

> Paul: This is one recipe for it [points to orange rectangle on the left] … if you get to 6 mL you actually have 2 of the recipe, like 2 times… it is like 3 times 2 mL and or 2 times 2 mL and you're able keep going up.

An exchange later, when the interviewer asked whether there was another way to show that lavender oil and rose oil had the same fragrance, Paul made the following strip diagram that indicates his use of the variable-parts perspective and partitive division (Fig. 13):

> Paul: … whatever you put in one of these boxes has to go into every single one of the other boxes. So say if you put 10 mL in the first box … then you would end up with like 30 total and 20 total.

Based on Paul's strip diagram, Amy explained below how her thinking from the multiple-batches perspective was different than Paul's thinking about the variable-parts perspective, which indicates an evidence for her keeping the two perspectives separate:

> Amy: I did the number of batches, like so if I did half a batch, 1/2 times this. But he did like okay so changing the batches you're changing your parts. So each part has got a half and then this would now instead of representing half a batch would be half a mL in each one, which ends up being the same because its 3/2 of lavender and one whole rose oil.

A moment later, the interviewer asked Amy to compare her own thinking with the multiple-batches perspective in Fig. 11 to her thinking with the variable-parts perspective for 21 mL of lavender oil and 14 mL of rose oil. In response to this question, she drew a strip diagram with 3 parts of lavender oil and 2 parts of rose oil and put 7 mL for each part. Thus, she was able to reason from the variable-parts perspective

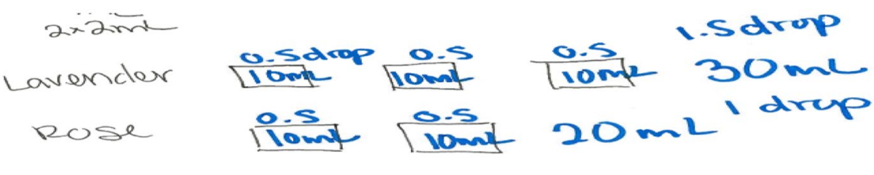

Fig. 13 Paul's drawing of a strip diagram

by keeping the number of parts fixed and varying the size of each part. These data provided additional evidence for Amy's keeping the two perspectives separate because she did not bring multiple-batches perspective thinking into her strip diagram.

The interviewer skipped Task 2 and moved to Task 3. Task 3 asked whether 5 mL of lavender oil and 3 mL of rose oil are in a 3-to-2 ratio; Paul found a common denominator for 5/3 and 3/2 and got the following equivalent fractions: 10/6 and 9/6. He concluded that 5 mL of lavender oil and 3 mL of rose oil were not in a 3-to-2 ratio because this mixture had a stronger lavender smell than the original mixture made in a 3 to 2 ratio. Other than this information, the data lacked evidence for his use of division and the perspective on ratios in this task. A moment later, Amy drew a ratio table (Fig. 14) and decided which of the given mixtures in Task 3 were in a 3-to-2 ratio by taking multiples of the original mixture. She said that "When I think of this, I'm thinking of simplifying 12 over 3 to be 4 batches, but I mean I guess that's not really simplifying."

These data indicate her use of quotitive division because she found the number of batches (or groups). Earlier in Task 1, she had considered the meaning of "simplifying" as partitive division. However, in Task 3, she viewed quotitive division as different than her meaning of simplifying, which indicates an evidence for her explicitly identifying different meanings for division.

Interview 2 for Amy and Paul

In response to Task 4, they quickly noticed that the 7-to-5 ratio would change if 4 grams of gold were added to the mixture. As opposed to Pair 1 and Pair 2, they could solve the task without any prompting from the interviewer. They explained clearly that 20/7 grams of copper would be needed to keep the same 7-to-5 ratio, with the following Amy's strip diagram (Fig. 15):

> Amy: I took my four grams that have been added to the pure gold, divided those among the seven parts, so that's four sevenths per part and then you need to multiply four sevenths times five … add that with the four grams of pure gold to keep it in the same mixture.
>
> Paul: … You have four grams … spread it over the seven so it would be four sevenths of a gram in each … to fill up every single one you need to multiply by five so that'd be twenty seventh grams total of copper that's needed to raise it up to the gold.

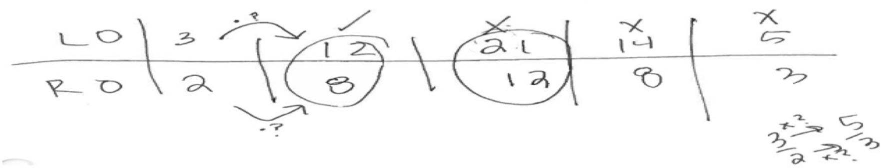

Fig. 14 Amy's drawing of a ratio table

Fig. 15 Amy's drawing of a strip diagram in the second interview

A moment later, the interviewer asked whether 20/7 grams of copper would depend on the amount of the initial mixture. Amy and Paul both agreed that regardless of the amount of the initial mixture, 20/7 grams of copper would be needed to keep the same 7-to-5 ratio. When the interviewer asked them to explain why their answer of 20/7 grams of copper makes sense regardless of the initial amount, Amy provided the following explanation that demonstrated her reasoning from the variable-parts perspective:

> Amy: Because if I had made like a punch or something where it was like seven parts of water and five parts of juice, I could have used like seven teaspoons of water and five teaspoons of juice, or like, seven gallons and five gallons … if I'm adding four more grams of water, I have to add five sevenths of that amount of juice to the mixture.

Discussion, conclusion, and implications

The results, based on six PSTs' interview data in a middle-grades teacher preparation program, suggest new, productive lines of research for ratios and proportional relationships. Past research has acknowledged the persistent difficulties teachers and PSTs have experienced identifying different meanings for division (e.g., Jong & Magruder, 2014; Simon, 1993). However, no study has examined how PSTs' understandings of division and their use of the two perspectives on ratios might support and constrain their formation of proportional relationships. By following the mathematical analysis developed by Beckmann and Izsák (2015) that extends parallels between different meanings for division and distinct perspectives on ratios, the present study examined PSTs' understandings of division and their reasoning about proportional relationships, which are key topics in the multiplicative conceptual field (Vergnaud, 1983, 1988).

All PSTs demonstrated the capacity to reason from the multiple-batches perspective, the variable-parts perspective, or both, when working on the Oil tasks in the first interviews. However, this capacity was not sufficient for some of them (e.g., Lisa, Amber) to differentiate the two perspectives in the first interviews, and to form appropriate proportional relationships in the second interviews. In particular, PSTs who could not keep the two perspectives separate tended to bring multiple-batches thinking into their strip diagram drawings that require reasoning from the variable-parts perspective. Therefore, one main result of this study was that PSTs who did not explicitly identify different meanings for division had trouble differentiating the two perspectives on ratios. Those teachers also had difficulty forming proportional relationships while solving the proportion tasks. On the other hand, PSTs who could maintain different meanings for division were also able to keep the two perspectives separate, and formed proportional relationships successfully on the proportion tasks. A robust understanding of proportional relationships includes explicitly identifying different meanings for division as well as reasoning about the two distinct perspectives separately.

In a recent study that examined the same six PSTs' reasoning about ratios, Ölmez (2016) reported that PSTs' reliance on multiplicative relationships instead of additive relationships supported their ability to keep the two perspectives separate on ratios. The present study also revealed that PSTs' understandings of division played an important role in differentiating the two perspectives on ratios. Therefore, it can be concluded that PSTs' ability to keep separate the two perspectives on ratios depends on their explicitly identifying different meanings for division, in addition to their reliance on multiplicative versus additive relationships. At the same time, it should be noted that differentiating the two perspectives on ratios might require having more facilities such as paying attention to units in proportion tasks. Future studies should investigate further which other topics in the multiplicative conceptual field, such as multiplication and fractions, might support this ability to keep the two perspectives separate and form proportional relationships. Similarly, future studies should also examine the extent to which PSTs' performance on a test involving problems about proportional relationships might be related to their reasoning from the two perspectives on ratios.

Another main result of this study was that PSTs mostly used the partitive meaning of division, which indicates that PSTs were more facile with partitive division than quotitive division. Given that students primarily use partitive division and learn to use quotitive division later through instruction (e.g., Fischbein et al., 1985) and their teachers and PSTs tend to rely on partitive division (e.g., Ball, 1990; Piel & Green, 2010; Simon, 1993; Timmerman, 2014), this result is consistent with past research. In the present study, while two PSTs used both types of division, three PSTs only used partitive division and one PST only used quotitive division. Considering the fact that those PSTs who only used partitive division were the ones who could not keep the two perspectives separate on ratios, PSTs' thinking only in terms of partitive meaning for division might limit those PSTs' ability to differentiate the perspectives on ratios and form proportional relationships. Teachers with both conceptual meanings for division will be better equipped to support their students' reasoning about proportional relationships. In addition to PSTs' use of partitive and quotitive division, three PSTs resorted to numeric division during interviews. In those situations, PSTs appeared to ignore the quantities and focused on the final result of the division operation. As similar to the findings of Simon (1993) and Piel and Green (2010), PSTs sometimes rely on procedural understanding of the division operation (i.e., obtaining the numeric answer) instead of focusing on the meaning of division.

Based on the present study, mathematics teacher educators should have the following two conceptual goals in their courses for reasoning about proportional relationships: (a) PSTs will develop meanings for both partitive and quotitive division and will demonstrate flexibility in translating between those meanings of division and (b) PSTs will reason from the two perspectives on ratios and make transitions between the perspectives by keeping them separate through appropriate wording and representation. To support such an awareness, it is important to provide PSTs with a range of experiences translating between different meanings for division and two perspectives on ratios. This implication is consistent with some recent studies (e.g., Beckmann et al., 2015; Orrill & Millett, 2021) arguing that teachers and PSTs might not readily connect critical understandings and could reason better after receiving instruction about them through teacher preparation programs and professional development.

Finally, an important implication of this study is that mathematics courses in middle-grades teacher preparation programs should be designed to deliberately support all the topics in the *multiplicative conceptual field* such as multiplication, division, fractions, ratios, and proportional relationships. In particular, the mathematics courses should address, among other key content, both meanings of division, both perspectives on ratios, and multiplicative relationships between quantities using double number lines and strip diagrams.

Future studies should continue to examine whether underlying facility of forming proportional relationships generalizes to explain performance of future teachers (and students) when reasoning about proportional relationships. Such studies should include a range of contextual differences that involve formation of proportional relationships.

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Declarations

Conflict of interest The author declares no competing interests.

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