Meccanica (2022) 57:3033–3049
https://doi.org/10.1007/s11012-022-01612-7

Approximate closed-form solutions for vibration of nano-beams of local/non-local mixture

Uğurcan Eroğlu · Giuseppe Ruta

Received: 17 August 2022 / Accepted: 3 November 2022 / Published online: 28 November 2022

Abstract This paper presents an approach to natural vibration of nano-beams by a linear elastic constitutive law based on a mixture of local and non-local contributions, the latter based on Eringen's model. A perturbation in terms of an evolution parameter lets incremental field equations be derived; another perturbation in terms of the non-local volume fraction yields the variation of the natural angular frequencies and modes with the 'small' amount of non-locality. The latter perturbation does not need to comply with the so-called constitutive boundary conditions, the physical interpretation of which is still debated. The possibility to find closed-form solutions is highlighted following a thorough discussion on the compatibility conditions needed to solve the steps of the perturbation hierarchy; some paradigmatic examples are presented and duly commented.

U. Eroğlu
Mechanical Engineering Department, MEF University, Istanbul, Turkey

U. Eroğlu (✉) · G. Ruta
Department of Structural and Geotechnical Engineering, University "La Sapienza", Roma, Italy
e-mail: erogluug@mef.edu.tr; ugurcaneroglu.guest@uniroma1.it

G. Ruta
National Group for Mathematical Physics of INdAM, Roma, Italy
e-mail: giuseppe.ruta@uniroma1.it

Keywords Nano-beams · Non-local elasticity · Perturbation methods · Constitutive boundary conditions · Natural vibration

1 Introduction

The origins of the theory of linear elasticity are reported, among others, in [1, 2] and [3], ch. 1, where the studies of Navier [4], Cauchy [5], Poisson [6] and Green [7] are recalled. In the beginning, matter was thought composed of molecules, which are equal infinitesimal body-points interacting by central forces depending only on their distance, with no particular inner organisation. Due to this, and since the radius of molecular activity is not appreciable at laboratory scales, such theories are dubbed *local* (often also *classical*). Gabrio Piola in mid 1800's suggested that material behaviour may be affected by different arrangements of molecules (see, e.g., [3], ch. 2, and [8]), and Bravais' studies on crystals (see [3], ch. 1) highlighted that molecules must be corpuscles with small, yet nonzero, extension: this paved the way for the Cosserats [9], Voigt [10] and Poincaré [11]. These studies foresee either an inner structure or non-central inner actions and opened the field of non-local continuum models. One of the first instances of the latter is in [12], where the convolution of a suitable kernel function with all the reactions of a foundation determines its local deformation. The researchers' desire of *understanding the world, rather than making it more*

comfortable [13] pushed the investigation on enriched models further: indeed, non-classical, non-local (in both sense of long-range and innerly structured) models can describe materials with heterogeneities and/or internal organization at a small scale with respect to external lengths, which affect their behaviour at larger scales. There is a very wide range of such materials and/or structural elements in interesting and up-to-date engineering applications, from carbon nanotubes and nanoshells [14] to masonry [15].

Non-classical linear elasticity uses continua endowed with enhanced constitutive relations or additional kinematic descriptors. Alternatively, material inner organization can be described by discrete models, as in molecular dynamic simulations for nano-sized elements [16] or in the limit analysis of brick walls at the macro-scale [17]. However, the numerical burden of these techniques, proportional to the dimensions of the considered assembly, may be unbearable for the computational power of our time. Continuum models,[1] on the other hand, may be used when the characteristic lengths are *sufficiently* larger than the smallest material unit of the body. Thus, the selection of an appropriate approach depends on the scale of the inner structure of the considered body or material, and on the intended outcomes of the problem. We will focus on continuum models for nano-sized beams, the dimensions of which are sufficiently greater than interatomic distances, which however still exhibit size-dependent behaviour.

Non-classical continua are labelled implicit/weak or explicit/strong [15, 18–20], depending on how size effects are accounted for. Implicit models add kinematic descriptors to those of local elasticity, plus their work-conjugates. In strong models the classical theory is endowed with *long-range interactions*,[2] leading to constitutive laws in terms of integral or integral-differential operators. Examples of implicit models are [9, 21, 22], while the like of [23–25] present explicit models. Covering all models and their possible variants is well beyond our scope; one may refer to [1, 18, 26–30] for a background and to [31–33] for reviews and comparisons of various non-local models.

Here we study nano-sized beams following Eringen's non-local elastic law, firstly presented in Kröner [23], Eringen and Edelen [24], and Eringen [34] and thoroughly in [25]. Its early applications focused on infinite or semi-infinite solids, for which a reduction of the original integral form to a differential one is possible [35]. The latter presents a Laplace operator and is relatively easy to handle from a mathematical point of view, which made it appreciated and used in many papers [36–43]. However, its application to finite domains fails, and evidence of paradoxical results is, e.g., in [44, 45]. Furthermore, Eringen's original integral form leads to integral-differential equations and requires satisfying additional boundary terms with respect to the standard natural and kinematic ones of local, classical elasticity. These additional terms are called *constitutive boundary conditions* [46–49], since they are of elastic nature, but still of uncertain physical interpretation. Indeed, they must account for the values of the possible kinematical constraints there, but the material must be actually *unaware* of their presence. Nevertheless, it is also reported that an exact solution may be available for some specific boundary and loading conditions, making the constitutive boundary conditions identically satisfied [50].

The mathematically problematic side of Eringen's constitutive law is somehow smoothened by its variant that considers the material as a two-phase mixture of local and non-local linear elastic behaviour [44, 51]. The material response is thus assumed to be a sum of suitably weighted local and non-local contributions, according to the usual law of mixtures in functionally graded, or composite, materials. Benvenuti and Simone [52] examined the axial behavior of nonlocal bars by the two-phase model of Eringen's. Simple bending and the elastica is examined by Bernoulli-Euler's theory by Wang *et al.* [53]. This research is extended to Timoshenko beam theory in [54]. Relations between discrete models and their continuum counterparts are investigated by Tuna *et al.* [55] using Eringen's two-phase model. An attempt on releasing the constitutive boundary conditions for static problems of bars and beams is presented by Eroğlu [56].

For free vibration, it is possible to obtain particular solutions that satisfy both balance and constitutive

[1] More precisely, *quasi-continua*, since distances shorter than the so-called *scale parameter* have no physical meaning [18].

[2] Here 'long' stands for 'with much larger radius of molecular activity than that of local elasticity'.

equations [57–60], but the need for constitutive boundary conditions remains, and such requirement of possibly spurious additional constraints, which are of uncertain physical interpretation, casts a shadow on these solutions [61–63]; this is also discussed in a recent review [64]. Therefore, it is important to look for physically motivated solutions to problems of simple, yet widely used, structural elements.

We deal with nano-sized beams and use a mixture of local and non-local elastic materials following Eringen's model, which is built upon well-accepted axioms of material response. In this sense, neither the beam model nor the constitutive model adopted in this contribution are original; the novelty of our work lies in the way we deal with the field equations of the problem of natural linear vibration for such nano-beams. Indeed, we look for an approximate solution of the field equations under the assumption that the contribution of the non-local portion in the constitutive equation is weighted by a *small* coefficient that assumes the role of a perturbation parameter, according to the schemes suggested, for instance, in [56, 65]. This *Ansatz* enables us to perform a formal power series expansion of the field functions in terms of the small perturbation parameter and yields a hierarchy of sets of field equations, one at each step of the formal expansion. As far as we know, this is a new way to approach nonlocal elastic problems and leads to a couple of meaningful pros. Indeed, the advantages of such a procedure are that: i) existence and uniqueness of the solutions are ensured for each set of the hierarchy, which behaves meekly from the point of view of mathematical complexity; and ii) by a suitable choice of the initial value of the perturbation parameter, it is possible to neglect the constitutive boundary conditions, which leave so many unanswered questions. This new approach, thus, gets rid of many of the theoretical and numerical difficulties met in previous investigations and represents a strong means to investigate a whole series of problems related to nanobeams. The main novelty of this work is extending the application of the perturbation approach, which was introduced in [56] for static problems, to linear vibration problems, the existence of the solutions of which must be further examined. By this method we will look for small-amplitude vibration natural angular frequencies of nano-beams versus the non-local fraction; the possibility of obtaining closed-form solutions, which is crucial in many applications, among which material identification procedures, will be investigated.

2 A direct one-dimensional beam model

In the 3D Euclidean ambient space we fix an origin and a Cartesian coordinate frame xyz, to which an orthonormal basis of vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is associated. We see a beam as a collection of equal prototype plane regions $\mathfrak{R}$, named beam cross-sections, orthogonally attached through their centroid to a portion of the z-axis; this segment, of length l from the origin, is named the beam axis. The portion of ambient space occupied by this construction represents the reference unstressed configuration $\mathcal{B}_0$ of the beam. If, as usual, we assume that the beam cross-sections undergo only rigid body motions, a different configuration $\mathcal{B}$ of the beam is reached by the displacement of the centroids of $\mathfrak{R}$, described by the vector field $\mathbf{u}(z)$, and the cross-sections rotation, described by the proper orthogonal tensor field $\mathbf{R}(z)$. Due to the assumed rigidity of the beam cross-sections, the kinematics (hence, the relevant dual work-conjugate quantities and, thus, all fields of mechanical interest) depend on the sole abscissa along the undeformed beam axis: the beam model so introduced is direct and one-dimensional. Henceforth, for simplicity of notation, the dependence of fields on z will be understood, hence omitted, if no confusion arises.

Strain is of local nature, defined as the difference between a generic change of shape and a rigid one; suitable strain measures in the actual configuration are [66]

$$\mathbf{E} = \frac{d\mathbf{R}}{dz}\mathbf{R}^{\mathrm{T}}, \quad \mathbf{e} = \frac{d}{dz}(z\mathbf{k} + \mathbf{u}) - \mathbf{R}\mathbf{k} \tag{1}$$

In Eq. (1) $\mathbf{E}$ is the skew-symmetric tensor field providing the curvatures of the beam axis in the actual configuration; $\mathbf{e}$ is the vector field describing how $\mathbf{k}$, coinciding with the unit tangent to the beam axis in $\mathcal{B}_0$, changes length and setting with respect to the cross-sections in $\mathcal{B}$. We are interested in small vibration about $\mathcal{B}_0$: the actual shapes, adjacent to it, are reached by 'small' displacements and rotations from $\mathcal{B}_0$. We may thus assume that all the fields of interest regularly depend on an evolution parameter and expand them in a formal power series up to the first order in the same parameter, neglecting the terms of

order higher than one. The zeroth order terms of $\mathbf{u}$ and $\mathbf{R}$ are the null vector and the identity tensor, as they are evaluated in $\mathcal{B}_0$.

If, for simplicity, we study vibration in the yz plane,[3] the axis displacement is decomposed as $\mathbf{u} = v\mathbf{i} + w\mathbf{k}$, and the cross-sections rotation is expressed in terms of a sole angle Ω about axes parallel to x. Then, $\mathbf{e}$ has two scalar components, denoted γ and ε, called shearing strain between axis and cross-sections and axial elongation, respectively; $\mathbf{E}$ has a sole component, denoted χ and called bending increment of curvature (or simply curvature in this case). Since we investigate adjacent configurations, the incremental, linearised strain measures are [66]

$$\varepsilon = \frac{dw}{dz}, \qquad \gamma = \frac{dv}{dz} + \Omega, \quad \chi = \frac{d\Omega}{dz} \tag{2}$$

where, with an abuse of notation, v, w, Ω are actually first-order increments with respect to the evolution parameter.

The mechanical balance in actual configuration may be obtained via the principle of virtual work [66] or by the so-called principle of solidification, requiring as a necessary condition the vanishing of the total force and torque for any neighbourhood of the considered geometrical model of the body [3]. Supposing the absence of *non-local residuals* [25], the local balance of force and torque in the actual shape reads

$$\frac{d\mathbf{n}}{dz} + \mathbf{q} = \mathbf{0}, \qquad \frac{d\mathbf{m}}{dz} + \frac{d(z\mathbf{k} + \mathbf{u})}{dz} \times \mathbf{n} + \mathbf{t} = \mathbf{0} \tag{3}$$

where $\mathbf{n}, \mathbf{m}$ are the inner force and couple and $\mathbf{q}, \mathbf{t}$ the outer force and couple densities, respectively. In our planar setting, we decompose the inner force as $\mathbf{n} = T\mathbf{j} + N\mathbf{k}$, labelling T, N as shearing and normal, respectively; the inner couple has the sole bending component M. To investigate balance in configurations adjacent to the referential one, similarly to what was done for strain measures, a formal series expansion up to the first order of an evolution parameter can be performed, yielding [66]

$$\frac{dN}{dz} + q_z = 0, \quad \frac{dT}{dz} + q_y = 0, \quad \frac{dM}{dz} - T + t_x = 0 \tag{4}$$

[3] It is enough to suppose geometric and material symmetry of the cross-sections about the y-axis, and all loads to lie on the yz plane.

With another abuse of notation, M, T, N, t_x, q_y, q_z are now first-order increments with respect to the evolution parameter of the components of inner action and of outer action densities with respect to the axes indicated by the subscripts.

In order to close the linear elastic problem we need one last set of relations between strain measures and internal actions. We suppose, without loss of generality and in accord with many works of the literature, that the considered beam is purely flexible. That is, we admit that the curvature χ is the only detectable strain measure, which thus constitutively prescribes the bending couple, its dual work-conjugate. As a consequence, the contact force components are pure constraint reactions. According to the law of mixtures, and assuming that ξ is the portion of non-local response following Eringen's simplified law for one-dimensional continua in the so-called strain-driven form, we have [57]

$$M = B\big[(1-\xi)\chi + \xi K * \chi\big] \tag{5}$$

where B is the bending stiffness of the cross-section; the non-local response is the convolution of the detectable strain with a so-called kernel K that accounts for the strain contribution of a finite neighbourhood of the considered point, or even of the whole domain. Among many alternatives, we use the following symmetric exponential [52, 54, 57]

$$K(\zeta, z) = K(|\zeta - z|) = \frac{1}{2\kappa}\exp\left(\frac{|\zeta - z|}{\kappa}\right), \qquad K * f = \int_0^L K(\zeta, z) f(\zeta) dz \tag{6}$$

where κ is the non-local characteristic length parameter, roughly providing a measure of long-range inner elastic actions, or, analogously, a kind of radius of sensible molecular activity, outside which the mechanical response quickly vanishes and is almost negligible.

2.1 Linear transverse vibration

If all fields of interest depend on time t in addition to the axial abscissa z, we may investigate 'small' transverse vibration with respect to the beam axis, where the sole outer action density is the relevant inertia; hence, the only non-zero distributed load q_y, by D'Alembert's principle, is [57]

$$q_y = -\mathfrak{m}\ddot{v} \tag{7}$$

Here $\mathfrak{m}$ is the cross-section mass (mass of the beam per unit length of the undeformed axis), and over-dots denote time derivatives. If only natural vibration consequent to a 'small' perturbation of the initial shape is considered, it is usual to assume that the field functions harmonically depend on time with natural angular frequency ω, i.e., $g(z,t) = g(z)\sin(\omega t + \phi)$ for any field g. The supposed pure flexibility implies the vanishing of shearing strain in Eq. $(1)_2$, so that the cross-section rotation depends on the slope of the deformed beam axis and the axis curvature depends on the second derivative of the transverse displacement of the beam by $(1)_3$; furthermore, the constitutive equation (5) and the local transverse balance (4) read

$$\begin{aligned} &\frac{dv}{dz} = -\Omega, \qquad (1-\xi)\frac{d\Omega}{dz} + \xi K * \frac{d\Omega}{dz} = \frac{M}{B}, \\ &\frac{dT}{dz} = -\mathfrak{m}\omega^2 v, \qquad \frac{dM}{dz} = T, \end{aligned} \tag{8}$$

where, for simplicity, only the spatial part of the fields is indicated, the time-harmonic part being understood.

In order to abstract from particular numerical values involved in the investigation, let us define the following non-dimensional quantities

$$\begin{aligned} &\bar{z} = \frac{z}{L}, \quad \bar{v} = \frac{v}{L}, \quad \bar{\kappa} = \frac{\kappa}{L}, \quad \bar{T} = \frac{TL^2}{B}, \\ &\bar{M} = \frac{ML}{B}, \quad \bar{\lambda} = \frac{\mathfrak{m}\omega^2 L^2}{B}, \quad \bar{N} = \frac{NL^2}{B} \end{aligned} \tag{9}$$

For typographic ease, even though this is another abuse of notation, henceforth we will consider only non-dimensional quantities and the overbars in Eq. (9) will be understood and hence omitted, except when confusion may arise.

3 Formal expansions

Since the material behaviour follows the law of a mixture, it is physically admissible that all the fields characterising the nano-beam response depend on the non-local fraction ξ. All fields may thus be formally expanded in power series around $\xi = \xi_0$, yielding [56]

$$\boldsymbol{\psi} = \sum_{i=0}^{\infty} \frac{(\xi-\xi_0)^i}{i!}\boldsymbol{\psi}_i, \qquad \lambda = \sum_{i=0}^{\infty} \frac{(\xi-\xi_0)^i}{i!}\lambda_i, \tag{10}$$

where the state vector $\boldsymbol{\psi}$ is defined as

$$\boldsymbol{\psi} = \{v, \Omega, T, M\}^T \tag{11}$$

and $g_i = (\partial g)/(\partial \xi)|_{\xi=\xi_0}$ for any field g. Truncating the series provides an approximation to the function of interest:

$$\boldsymbol{\psi} \approx \sum_{i=0}^{N} \frac{(\xi-\xi_0)^i}{i!}\boldsymbol{\psi}_i, \qquad \lambda \approx \sum_{i=0}^{N} \frac{(\xi-\xi_0)^i}{i!}\lambda_i \tag{12}$$

A suitable selection for the reference value of the perturbation parameter around which we seek approximations is $\xi_0 = 0$, corresponding to local elasticity. Thus, such procedure is suitable to investigate the effect of 'small' non-locality fractions seen as a perturbation of a fully local linear elastic material behaviour.

Expanding the field functions and the eigenvalues into formal power series as described above provides a hierarchy of equations in terms of the fraction coefficient ξ. Limiting the formal power series to the second order in ξ provides the following three sets of equations at different levels of non-locality for free vibration problems; details about the passages are reported in the Appendix for sake of space. At the zeroth order in the expansion in powers of ξ we have

$$\begin{aligned} &\frac{dv_0}{dz} = -\Omega_0 \quad \frac{d\Omega_0}{dz} = M_0 \\ &\frac{dT_0}{dz} = -\lambda_0 v_0 \quad \frac{dM_0}{dz} = T_0 \end{aligned} \tag{13}$$

where the subscript recalls the order of the formal expansion. These equations are identical to that of local beams, since we choose $\xi_0 = 0$, i.e., absence of the non-local portion in the material mixture. The set of field equations corresponding to the first order term of the expansion in ξ, i.e., 'small' fractions of non-local linear elastic behaviour, is

$$\begin{aligned} &\frac{dv_1}{dz} = -\Omega_1 \qquad \frac{d\Omega_1}{dz} = M_1 + \frac{d\Omega_0}{dz} - K * \frac{d\Omega_0}{dz} \\ &\frac{dT_1}{dz} = -\lambda_0 v_1 - \lambda_1 v_0 \qquad \frac{dM_1}{dz} = T_1 \end{aligned} \tag{14}$$

The set of field equations corresponding to the second order term of the expansion in ξ, i.e., 'moderate' fractions of non-local linear elastic behaviour, is

$$\frac{dv_2}{dz} = -\Omega_2, \quad \frac{d\Omega_2}{dz} = M_2 + 2\frac{d\Omega_1}{dz} - 2K * \frac{d\Omega_1}{dz}$$
$$\frac{dT_2}{dz} = -\left(2\lambda_1 v_1 + \lambda_0 v_2 + \lambda_2 v_0\right), \quad \frac{dM_2}{dz} = T_2 \tag{15}$$

The mechanical motivation of choosing only the first three steps of the perturbation hierarchy, represented by Eqs. (13)–(15), lies in the will to consider first a fully local material (zeroth order, Eq. (13), in order to make comparisons with well established results of the literature), then to investigate how slowly increasing fractions of nonlocal material affect the actual behaviour. In this sense, we consider a ''small' (in the sense of perturbation methods) nonlocal fraction (first order, Eq. (14)), then a slightly more relevant nonlocal fraction (second order, Eq. (15)). These two steps represent another point of novelty of this contribution, in that the perturbation approach lets us investigate these problems without the need to resort to the dubious constitutive boundary conditions.

Eq. (13)–(15) show usual patterns of perturbation expansions: the differential operator is the same for all sets of the hierarchy, but while the first set is homogeneous, higher-order ones include non-homogeneities which depend on the solutions of the sets of equations at previous orders. These systems may be written in the following matrix form

$$\mathbf{D}\boldsymbol{\psi}_0 = \mathbf{0}$$
$$\mathbf{D}\boldsymbol{\psi}_k = \mathbf{f}_k \quad k = 1, 2, \ldots, \quad \mathbf{D}(\cdot) = \mathbf{I}\frac{d(\cdot)}{dz} - \mathbf{A}(\cdot) \tag{16}$$

where: the forcing term $\mathbf{f}_k$ at the k-th order of the expansion in ξ is written in terms of the solutions of the preceding sets of equations of the hierarchy, as usual in perturbation techniques; $\mathbf{I}$ is the 4×4 identity matrix; and the 4×4 matrix $\mathbf{A}$ is as follows

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\lambda_0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{17}$$

Please remark that Eq. (16) describe only the bulk behaviour, i.e., they do not take into account boundary conditions, which must be added.

The solution of each set of differential equations (16) can be found in integral form [67]

$$\boldsymbol{\psi}_0 = \mathbf{Y}(z, z_0)\boldsymbol{\psi}_{0,0}, \quad \boldsymbol{\psi}_k = \mathbf{Y}(z, z_0)\boldsymbol{\psi}_{k,0} + \mathbf{g}_k(z)$$
$$\mathbf{g}_k(z) = \mathbf{Y}(z, z_0)\int_{z_0}^{z} \mathbf{Y}(z_0, \zeta)\mathbf{f}_k(\zeta)d\zeta \tag{18}$$

where $\mathbf{Y}(z, z_0)$, labelled *fundamental* or *transfer* matrix, is also dubbed *matricant*; it provides the state vector $\boldsymbol{\psi}_k$ at a generic point z once the same vector, denoted $\boldsymbol{\psi}_{k,0} = \boldsymbol{\psi}_k(z_0)$, is known at a point z_0. That is, the complete solution expressed by Eq. (18) requires the specification of some boundary conditions, which is to be expected since, as already remarked, we considered no such conditions so far.

When the matrix $\mathbf{A}$ has uniform coefficients with respect to z, as in our case (see Eq. (17)), the transfer matrix has the very simple expression [67]

$$\mathbf{Y}(z, z_0) = \exp\left(\mathbf{A}(z - z_0)\right) \tag{19}$$

For completeness, the components of the matricant for the case of our investigation are reported in the Appendix.

When $\mathbf{A}$ has variable coefficients, approximate methods to determine the matricant are available; see, for example, [68]. Thus, a general solution to the system of bulk equations (16) can be always found, at least approximately, in terms of $\boldsymbol{\psi}_{k,0}\ \forall k$. However, when the bulk equations are supplemented by kinematic or natural boundary conditions at the first step of the perturbation equations, a fully homogeneous system turns out, and non-trivial solutions exist only if the operator expressing the boundary conditions is singular. Then, since the bulk and the boundary operators (be they differential or algebraic) are the same at each step of the hierarchy derived from the perturbation procedure in terms of the non-local fraction coefficient, a solvability condition on the forcing term $\mathbf{f}_k$ shall be provided [65].

For the solvability of the sets of equations at every step of the perturbation hierarchy, it is fundamental to remark that $\mathbf{A}$ is not symmetric, i.e., $\mathbf{A} \neq \mathbf{A}^{\mathrm{T}}$. Then, the algebraic-differential operator $\mathbf{D}$ at the left-hand side of every set of equations in Eq. (16) is not self-adjoint with respect to the usual inner product for vector fields with real-valued components

$$\langle \mathbf{a}, \mathbf{b} \rangle = \int_0^1 \mathbf{a}^{\mathrm{T}}\mathbf{b}\, dz \tag{20}$$

where the non-dimensional spatial domain is the real interval [0, 1], see Eq. (9). Owing to this, the

orthogonality of the natural solutions Eq. (18) is awkward to verify, which makes the compatibility of the k-th set of the hierarchy Eq. (16) complicated to evaluate. Thus, in order to avoid unnecessary mathematical artifices, it is suitable to look for a slightly different strategy, which we believe to be another original point of this contribution. To start with, it is known that it is possible to reduce the set Eq. (8) to a single ordinary differential equation in terms of any of the meaningful field functions, leading to

$$\begin{aligned} &(1-\xi)f'''' + \xi\left(K * f^{(i)}\right)^{(4-i)} - \lambda f = 0, \\ &f = v \Rightarrow i = 1, \quad f = \Omega \Rightarrow i = 2, \\ &f = T \Rightarrow i = 3,\ f = M \Rightarrow i = 4 \end{aligned} \tag{21}$$

where primes stand for derivatives with respect to the non-dimensional axial abscissa, and the counter i between parentheses in the superscripts indicates another order of spatial derivative, depending on the chosen representative field function. The reason to prefer this formulation instead of the four first-order equations (16) is that while the orthogonality of real-valued scalar functions as the integral of their product over a finite domain is always possible, provided some very mild integrability conditions, the same is not warranted for vector fields like in Eq. (16): indeed, here $\mathbf{A}$ being non-self-adjoint plays a crucial role.

The choice of one of the fields of interest, hence of Eq. (21) as representative of the set Eq. (8) seems immaterial at a first sight; however, this is not true, due to the aspect of the corresponding boundary conditions.

As a starting point, let us consider the last of Eq. (21), which is in terms of the bending couple. The first three terms of the perturbation expansion of these bulk equations in terms of ξ are

$$\begin{aligned} &M_0'''' - \lambda_0 M_0 = 0 \\ &M_1'''' - \lambda_0 M_1 = M_0'''' - K * M_0'''' + \lambda_1 M_0 \\ &M_2'''' - \lambda_0 M_2 = 2M_1'''' - 2K * M_1'''' - 2\lambda_1 M_1 - \lambda_2 M_0 \end{aligned} \tag{22}$$

Again, we note the usual pattern of perturbation expansions.

3.1 Boundary conditions

The key point with the boundary conditions for this mixture of local and non-local elastic fractions is that, having operated a perturbation approach in terms of the non-local fraction ξ about $\xi = 0$, all quantities are evaluated at $\xi = 0$, i.e., for a local elastic medium. Indeed, the view proposed in this paper is to look at the non-local fraction as a perturbation of a wholly local elastic behaviour. This is crucial because in a wholly local elastic medium there is no need to fulfill the much debated constitutive boundary conditions, which do not enter neither the weak nor the strong mathematical formulation of the problem, and one shall account only for the standard kinematic or natural ones. Thus, recalling that all fields are actually first-order increments with respect to ξ, the usual boundary conditions are

$$\begin{aligned} &\text{fixed} \quad v = 0 \ \ \Omega = 0, \\ &\text{pinned} \quad v = 0 \ \ M = 0, \\ &\text{free} \quad T = 0 \ \ M = 0. \end{aligned} \tag{23}$$

Indeed, even in movable constraints the first-order increment of the displacement component vanishes, and natural boundary conditions for perfect constraints foresee vanishing reactions for all compatible first-order permitted motions. When the boundary values of the field functions are replaced with their series expansions, the boundary conditions being homogeneous imply that their expansions at each k are homogeneous. This is of practical importance, as it means that the boundary terms in the weak formulation of the problem are identical at all orders of the hierarchy.

On the other hand, if one of Eq. (21) is to be used, the same boundary conditions are not necessarily written by means of an identical operator at each order. Indeed, when the first of Eq. (21) is chosen, for example, a free end provides the following well-known boundary conditions at the zeroth order in ξ in terms of the non-dimensional transverse displacement

$$v_0'' = 0, \qquad v_0''' = 0. \tag{24}$$

However, at the first order in ξ the same conditions read

$$v_1'' - v_0'' + K * v_0'' = 0, \quad v_1''' - v_0''' + (K * v_0'')' = 0 \tag{25}$$

Thus, even if the bulk operator at the left-hand side of the hierarchy of equations (22) is the same at all steps, the boundary operators are not. This means that a family of orthogonal solutions at the zeroth order in ξ, satisfying bulk and boundary conditions at this step, cannot be an eigenvector basis for higher orders in ξ, since they do not necessarily satisfy the boundary conditions at these higher orders. However, this does not occur when the last of Eq. (21), which is in terms of the bending couple, is chosen. Indeed, Eq. (8) imply that, when considering harmonic vibration, not only the shearing force, but also the kinematic descriptors can be expressed in terms of the bending couple:

$$T = M', \quad v = -\frac{T'}{\mathfrak{m}\omega^2} = -\frac{M''}{\mathfrak{m}\omega^2}, \quad \Omega = -\frac{M'''}{\mathfrak{m}\omega^2} \tag{26}$$

Eq. (26), which is expressed by dimensional quantities, has a straightforward non-dimensional parallel; we omit the passages and keep the same symbols, with an abuse of notation but with the aim of saving space. Eq. (26) can be easily submitted to the usual formal ξ-power series expansion, so that we can obtain the terms that enter the boundary conditions at the various orders of the hierarchy of bulk equations Eq. (16). We can thus see that the boundary conditions in terms of the bending couple are

$$\begin{array}{llll} M_0 = 0, & M_0' = 0, & M_1 = 0, & M_1' = 0, \\ M_0 = 0, & M_0'' = 0, & M_1 = 0, & M_1'' = 0, \\ M_0'' = 0, & M_0''' = 0, & M_1'' = 0, & M_1''' = 0 \end{array} \tag{27}$$

for a free, simply supported, and a clamped end, respectively. That is, in any case the boundary conditions expressed in terms of the bending couple at each order are represented by the same operator acting on the corresponding field; hence, higher-order bending couples can be projected onto the basis of the zeroth-order eigenspace of bending couples.

We must remark that the simple expressions in Eq. (26), (27) hold only because we choose to investigate 'small' transverse translational vibration of the cross-sections of purely flexible, Euler-Bernoulli beams. Had we considered shearing strain in addition to bending curvature to represent the deformation of the beam, and rotary inertia of its cross-sections in addition to that in translation to represent the mass distribution properties of the beam, the boundary conditions would have been strongly coupled and not as simple as in Eqs. (26), (27). It is apparent that a starting investigation like this one suitably considers the most simple setting, in order to catch the essentials of the considered behaviour.

3.2 Fredholm compatibility condition

Imposing the boundary conditions (23) at each beam end provides four algebraic equations in terms of the unknown state vector $\boldsymbol{\psi}_{k,0}$. At the zeroth-order in ξ, implying local elasticity, these conditions in matrix form read

$$\mathbf{H}(\lambda_0)\boldsymbol{\psi}_{0,0} = \mathbf{0} \tag{28}$$

In order to have non-trivial solutions for $\boldsymbol{\psi}_{0,0}$, and, subsequently, for the state vector $\boldsymbol{\psi}_0$ at all points of the beam according to Eq. (18), the 4×4 matrix $\mathbf{H}(\lambda_0)$ shall be singular, which equals to search the values of λ_0 for which

$$\det(\mathbf{H}(\lambda_0)) = 0 \tag{29}$$

whence $\operatorname{rank}(\mathbf{H}(\lambda_0)) < 4$; thus, (29) provides the family of the zeroth-order eigenvalues λ_0 for 'small' vibration to exist.

Moving up to higher orders in ξ, we end up with a non-homogeneous system of algebraic equations in terms of the higher-order state vector at a given point z_0:

$$\mathbf{H}(\lambda_0)\boldsymbol{\psi}_{k,0} = \mathbf{b}_k(\lambda_0, \lambda_1, \dots, \lambda_k) \tag{30}$$

It is important that, since the matricant is the same at each order, the matrix $\mathbf{H}$ in (30) is identical to that in (28). The non-homogeneous forcing term $\mathbf{b}_k$, which is a function of all eigenvalues up to the k^{th} order, consists of elements of the kind of $\mathbf{g}_k$ in Eq. $(18)_2$ evaluated at $z = 0$ or $z = L$. Since $\mathbf{H}(\lambda_0)$ is singular, the condition for the existence of non-trivial $\boldsymbol{\psi}_{k,0}$ and, hence, of non-trivial solutions at the k-th order of the formal ξ-power series expansion, is that the system (30) be compatible. According to the well-known theorem by Rouché and Capelli, this equals to requiring the rank of $\mathbf{H}$ to be equal to that of the matrix

obtained by appending the column of known terms $\mathbf{b}_k$ to $\mathbf{H}$. This condition equals to resort to the vectors $\tilde{\boldsymbol{\psi}}$ such that

$$\left[\mathbf{H}(\lambda_0)\right]^{\mathrm{T}}\tilde{\boldsymbol{\psi}} = \mathbf{0} \tag{31}$$

that is, the vectors of the null space of the transpose of $\mathbf{H}(\lambda_0)$, which is well-defined since the matrix is real-valued and in a finite dimensional vector space. The vectors $\tilde{\boldsymbol{\psi}}$ are not trivial if $\mathbf{H}(\lambda_0)$ is singular (implying that its transpose is singular as well), as required by Eq. (29). Then, taking the ordinary dot product of both sides of Eq. (30) with $\tilde{\boldsymbol{\psi}}$ and operating standard operations of matrix algebra yields

$$\tilde{\boldsymbol{\psi}} \cdot \mathbf{b}_k = 0, \tag{32}$$

named *Fredholm's compatibility condition*, which ensures that non-trivial state vectors $\boldsymbol{\psi}_{k,0}$ exist, though depending on a normalisation parameter [65]. Eq. (32) enables us to find higher-order eigenvalues, i.e., the variation of natural angular frequencies with respect to those of local elasticity after the formal expansion in terms of the non-local fraction coefficient ξ. If we pose the bulk problem with the fourth-order ODE in terms of the bending couple, Eq. (21), and then perform the ξ-expansion, Eq. (22), the differential operator of the zeroth-order equation

$$D(\cdot) = \frac{d^4}{dz^4}(\cdot) - \lambda_0(\cdot) \tag{33}$$

is self-adjoint with respect to the scalar product (20). Then, the family of eigensolutions $\{M_0^j\}$ associated with the j-th eigenvalue λ_0^j is an orthogonal basis (the property of orthogonality of the eigenmodes expressed in terms of the bending couple is proved in the Appendix to save space here) and the j-th solution of the first-order equation Eq. $(22)_1$ has expression

$$M_1^l = M_0^l + \sum_{l\neq j} b_{lj} M_0^j \tag{34}$$

since the boundary conditions are identical at each order. Then, the ξ-first-order equation in Eq. (22) becomes

$$\sum_{l\neq j} b_{lj}\lambda_0^j M_0^j - \lambda_0^l \sum_{l\neq j} b_{lj} M_0^j = \lambda_0^l M_0^l - K*(\lambda_0^l M_0^l) + \lambda_1^l M_0^l \tag{35}$$

Multiplying both sides with M_0^m and integrating over the domain provides the orthogonality condition for $l = m$

$$\int_0^1 M_0^m\left(\lambda_0^m M_0^m - \lambda_0^m K * M_0^m + \lambda_1^m M_0^m\right)dz = 0 \tag{36}$$

Solving (36) for λ_1^m provides the first-order increment of the m-th eigenvalue with respect to the non-local elastic fraction *in closed form* (details are in the Appendix):

$$\lambda_1^m = \lambda_0^m \frac{\left\langle M_0^m, K * M_0^m\right\rangle - \left\langle M_0^m, M_0^m\right\rangle}{\left\langle M_0^m, M_0^m\right\rangle} \tag{37}$$

It is barely necessary to remark how important closed-form solutions in view of applications are. With Eq. (37), the first-order approximation to the m-th eigenvalue is

$$\tilde{\lambda}^m = \lambda_0^m + \xi\lambda_1^m \tag{38}$$

where a tilde is used to prevent abuse of notations.

4 Illustrative examples

Here we present the results of the procedure presented in the previous sections for the paradigmatic cases of the simply supported and the clamped-free beam. For each of these two cases, we will understand that the bulk problem is expressed in terms of the single Eq. (21), formally expanded in ξ as in Eq. (22). Then, we provide the relevant expression of the boundary operator $\mathbf{H}(\lambda_0)$, find the natural angular frequencies as well as the natural modes, and, by resorting to the solvability condition at the first order, we provide the closed-form expressions for the first-order increment of the natural angular frequencies for 'small' non-local volume fractions. For sake of space and just to show the features of the proposed technique, we limit to these two steps, i.e., we do not look for the first-order incremental modes and we do not impose the solvability condition for the third step

of the hierarchy, Eq. $(22)_3$ (second order in ξ). Indeed, from the point of view of the applications, being able to evaluate the variations of the natural angular frequencies in closed form already provides a remarkable means of describing the behaviour of this kind of nano-beams.

4.1 Simply supported beam

The boundary conditions at the zeroth- and first-order ξ -expansion are expressed by Eq. $(27)_1$. We may then evaluate $\mathbf{H}$ and perform the subsequent steps of the procedure.

4.1.1 Zeroth-order solution

The pinned boundary conditions at both ends of the beam lead to the following non-zero components of the coefficients matrix $\mathbf{H}(\lambda_0)$

$$\begin{aligned}
H_{11} &= 1 = H_{24},\\
H_{31} &= \frac{\cos\lambda_0^{1/4}+\cosh\lambda_0^{1/4}}{2} = H_{44},\\
H_{32} &= -\frac{\sin\lambda_0^{1/4}+\sinh\lambda_0^{1/4}}{2\lambda_0^{1/4}} = -H_{43},\\
H_{33} &= \frac{\sin\lambda_0^{1/4}-\sinh\lambda_0^{1/4}}{2\lambda_0^{3/4}} = -\frac{H_{42}}{\lambda_0},\\
H_{34} &= \frac{\cos\lambda_0^{1/4}-\cosh\lambda_0^{1/4}}{2\lambda_0^{1/2}} = \frac{H_{41}}{\lambda_0}
\end{aligned} \tag{39}$$

As already discussed, imposing the determinant of $\mathbf{H}(\lambda_0)$ to vanish in order to find non-trivial solutions for the natural vibration of the considered nano-beams provides a non-linear algebraic equation in terms of λ_0, which leads to

$$\lambda_0^n = n^4\pi^4, \quad \boldsymbol{\psi}_{0,0} = \Omega_{0,0}\left\{0, 1, -n^2\pi^2, 0\right\}^T \tag{40}$$

where $n \in \mathbb{Z}^+$ is the mode number. By recalling (9), it is apparent that the natural angular frequencies in $(40)_1$ are exactly those for local elastic simply supported beams. In addition, we now can insert the state vector $\boldsymbol{\psi}_{0,0}$ in $(40)_2$ into Eq. (18), keeping into account Eq. (19) and (17), and obtain the natural modes, which are the well-known sines.

The vector $\tilde{\boldsymbol{\psi}}$ satisfying Eq.(31) turns out to be expressed in terms of a free parameter of normalisation p as follows

$$\tilde{\boldsymbol{\psi}} = p\left\{-(-1)^n n^2\pi^2, -(-1)^n, n^2\pi^2, 1\right\}^T. \tag{41}$$

4.1.2 First-order solution

The non-zero components of the vector $\mathbf{f}_1$, representing the non-homogeneous part of the first-order differential equation (22), are

$$\begin{aligned}
f_{1,2} &= \omega_{0,0}\frac{n^2\pi^2\kappa\left[e^{-\frac{z}{\kappa}}-(-1)^n e^{\frac{z-1}{\kappa}}-2n\pi\kappa\sin(n\pi z)\right]}{2\left(n^2\pi^2\kappa^2+1\right)}\\
f_{1,3} &= \Omega_{0,0}\frac{\lambda_1\sin(\pi z)}{\pi}
\end{aligned} \tag{42}$$

The full solution of the set of field equations at the first-order of the formal ξ-expansion and the non-homogeneous part of the system in (30) for the same step are too long to be reported and thus are omitted for brevity. The condition of orthogonality Eq. (32) for this problem leads to

$$\begin{aligned}
&\frac{p\Omega_{0,0}}{2n\pi\left(1+\pi^2\kappa^2n^2\right)^2}\Big\{(-1)^{n+1}\Big[\lambda_1^n\left(\pi^2\kappa^2n^2+1\right)^2\\
&+\pi^6\kappa^2n^6\left(\pi^2\kappa^2n^2-2\kappa+1\right)\Big]-e^{-\frac{1}{\kappa}}2\pi^6\kappa^3n^6\Big\}=0
\end{aligned} \tag{43}$$

In order to have non-trivial $\tilde{\boldsymbol{\psi}}, \boldsymbol{\psi}_0$, Eq. (43) yields

$$\lambda_1^n = -\frac{\pi^6\kappa^2n^6\left\{1+\kappa\left[\pi^2\kappa n^2+2e^{-\frac{1}{\kappa}}(-1)^n-2\right]\right\}}{\left(\pi^2\kappa^2n^2+1\right)^2} \tag{44}$$

which provides the slope of the curve λ vs. ξ at $\xi = 0$, that is, the linear approximation to the variation of the fundamental eigenvalues with the nonlocal fraction of the material response in the neighbourhood of a fully local linear elastic material. We remark that using Eq. (37) provides the same result of Eq. (44), provided that

$$M_0^n(z) = -n\pi\Omega_{0,0}\sin n\pi z, \tag{45}$$

well-known for locally elastic Euler-Bernoulli beams.

Table 1 Coefficients for clamped-free beam

	λ_0			
	12.3624	485.519	3806.55	14617.3
x_1	2.15558	− 4.1491	3.9938	− 4.00027
x_2	3.13197	− 1.73574	1.0177	− 0.727592
x_3	1.13765	− 0.181534	0.0648327	− 0.0330847
x_4	− 12.3624	− 485.519	− 3806.55	− 14617.3
x_5	3.78953	45.7117	123.203	241.82
x_6	−0.858244	13.2943	45.9042	98.9182
x_7	− 2	− 2	− 2	− 2
x_8	1	1	1	1
x_9	0.30653	0.0941502	0.0323661	0.0165434
x_{10}	0.0694239	− 0.0273816	−0.0120593	− 0.00676721
x_{11}	− 0.161781	− 0.0041193	-5.25411×10^{-4}	− 0.000136824

4.2 Cantilevered beam

4.2.1 Zeroth-order solution

We assume the clamp at the cross-section corresponding to $z=0$, while that at $z=1$ is free. Then, the coefficients matrix $\mathbf{H}(\lambda_0)$ has the following non-zero components

$$\begin{aligned}
&H_{11}=1=H_{22},\\
&H_{31}=-\frac{\lambda_0^{3/4}}{2}\left(\sinh\lambda_0^{1/4}+\sin\lambda_0^{1/4}\right)=-\lambda_0 H_{43},\\
&H_{32}=\frac{\lambda_0^{1/2}}{2}\left(\cosh\lambda_0^{1/4}-\cos\lambda_0^{1/4}\right)=-H_{41},\\
&H_{33}=\frac{1}{2}\left(\cosh\lambda_0^{1/4}+\cos\lambda_0^{1/4}\right)=H_{44}\\
&H_{34}=\frac{\lambda_0^{1/4}}{2}\left(\sinh\lambda_0^{1/4}-\sin\lambda_0^{1/4}\right)=H_{42}
\end{aligned}\tag{46}$$

The characteristic equation and the corresponding non-trivial solutions for the initial state vector are

$$\begin{aligned}
&\tfrac{1}{2}\left(1+\cos\lambda_0^{1/4}\cosh\lambda_0^{1/4}\right)=0\\
&\boldsymbol{\psi}_{0,0}=M_{0,0}\{0,0,\alpha,1\},\\
&\alpha=-\frac{\lambda_0^{1/4}\left(\cos\lambda_0^{1/4}+\cosh\lambda_0^{1/4}\right)}{\sin\lambda_0^{1/4}+\sinh\lambda_0^{1/4}}
\end{aligned}\tag{47}$$

It is apparent that Eq. $(47)_1$ provides an implicit expression for the eigenvalues of the wholly local beam that coincides with the well-known results of the literature.

Eq. (47) implies that the vector $\tilde{\boldsymbol{\psi}}$ satisfying Eq. (31) is given by

$$\begin{aligned}
&\tilde{\boldsymbol{\psi}}=\{a_1,a_2,a_3,1\}p,\\
&a_1=-\frac{\lambda_0^{1/2}\sin\lambda_0^{1/4}\sinh\lambda_0^{1/4}}{\cos\lambda_0^{1/4}+\cosh\lambda_0^{1/4}},\quad a_3=\frac{1}{\alpha},\\
&a_2=(\lambda_0^{1/4})\frac{\sin\lambda_0^{1/4}\cosh\lambda_0^{1/4}-\cos\lambda_0^{1/4}\sinh\lambda_0^{1/4}}{\cos\lambda_0^{1/4}+\cosh\lambda_0^{1/4}}.
\end{aligned}\tag{48}$$

where, as in Eq. (41), p is a free parameter of normalisation.

The non-homogeneous parts of both systems of bulk differential equations and of boundary algebraic conditions are too long to be reported and will be omitted for sake of space, since they add no particular interesting results to our investigation.

On the other hand, we present the closed-form expression for the first incremental eigenvalue λ_1 resulting from imposing the solvability of the field equations at this step. This expression is given in terms of 11 numerical coefficients $x_1, x_2, \ldots, x_{11}$, deriving from the finite integration involved in the solvability condition Eq. (37), and the values of which are reported in Table 1 for the first four zeroth order (i.e., wholly local elastic) eigenvalues.

$$\begin{aligned}
\lambda_1^n=&\frac{1}{\left(1/\lambda_0^n-\kappa^4\right)^2}\Big[e^{-\frac{1}{\kappa}}\kappa^3\left(x_1\kappa^2+x_2\kappa+x_3\right)\\
&+\kappa\left(x_4\kappa^7+x_5\kappa^6+x_6\kappa^5+x_7\kappa^4+x_8\kappa^3\right.\\
&\left.+x_9\kappa^2+x_{10}\kappa+x_{11}\right)\Big]
\end{aligned}\tag{49}$$

Table 2 Comparison of the values for $\tilde{\lambda}^1$; BC= boundary conditions, SS= simply supported, CF= clamped-free

BC	κ		ξ	
			0.25	0.5
SS	0.05	[58]	9.84255	9.81485
		[57]	9.84255	9.81485
		Present	9.84276	9.81584
	0.1	[58]	9.77714	9.67978
		[57]	9.77714	9.67978
		Present	9.77854	9.68661
CF	0.05	[58]	3.4705	3.41739
		[57]	3.4705	3.41739
		Present	3.47298	3.4294
	0.1	[58]	3.42887	3.32818
		[57]	3.42887	3.32818
		Present	3.4328	3.34751

5 Numerical results and discussion

After having provided the explicit expressions for both addends λ_0, λ_1 of the formal first-order expansion of the eigenvalues of the natural transverse vibration for the two paradigmatic schemes of simply supported and clamped-free beam, we provide some particular numerical results for their actual sum using Eq. (38).These values are obtained by assuming both different amounts of the non-local fraction ξ in the mixture representing the linear elastic response of the nano-beam and of the non-local characteristic length parameter κ (we recall that we are operating with non-dimensional quantities), and are reported in Table 2, together with the corresponding numerical values provided in some recent literature.

It is immediately apparent that our perturbation approach can reply almost exactly the results found in the literature, with a somehow slight stiffening effect due to the coarse truncation of the formal series expansion at the first order. Yet such a coarse truncation well suffices to provide reliable results, and, once again, those results come from closed-form expressions. It is also highly remarkable that the results of our linear perturbation approach are reliable even for moderate non-local fractions, which is not to be taken for granted in advance.

Once found the zeroth- and first-order eigenvalues of natural vibration with respect to the formal ξ -expansion, it is interesting to find the corresponding non-dimensional natural angular frequencies, which are directly linked to the square root of the eigenvalues. Then, it is interesting to check how the first-order natural angular frequency is related to the corresponding zeroth-order one, which represents the known value for wholly local elastic beams. This also has a practical importance in material modelling and identification procedures; the closed-form

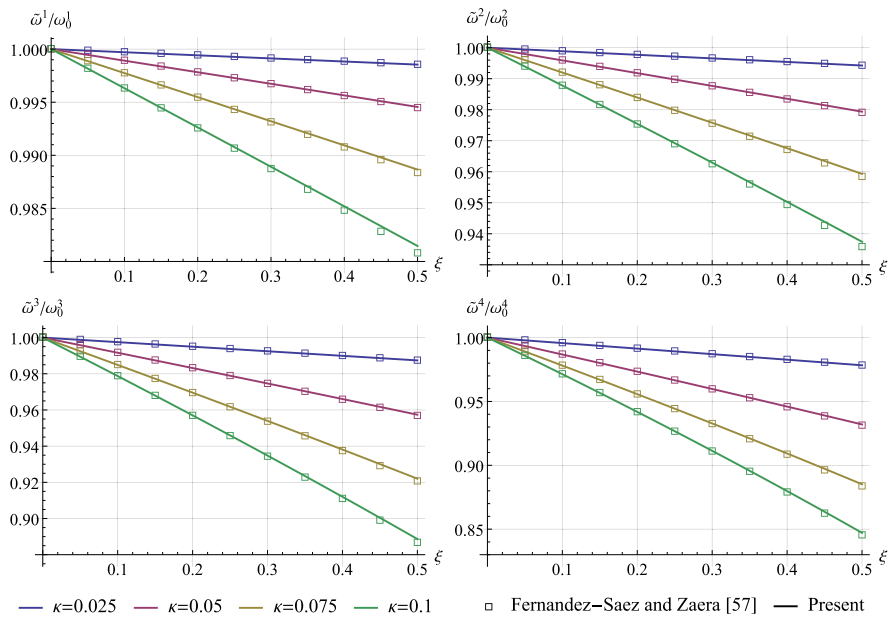

Fig. 1 Variation of the first-order ξ-expansions for the first four eigenvalues of a simply supported beam

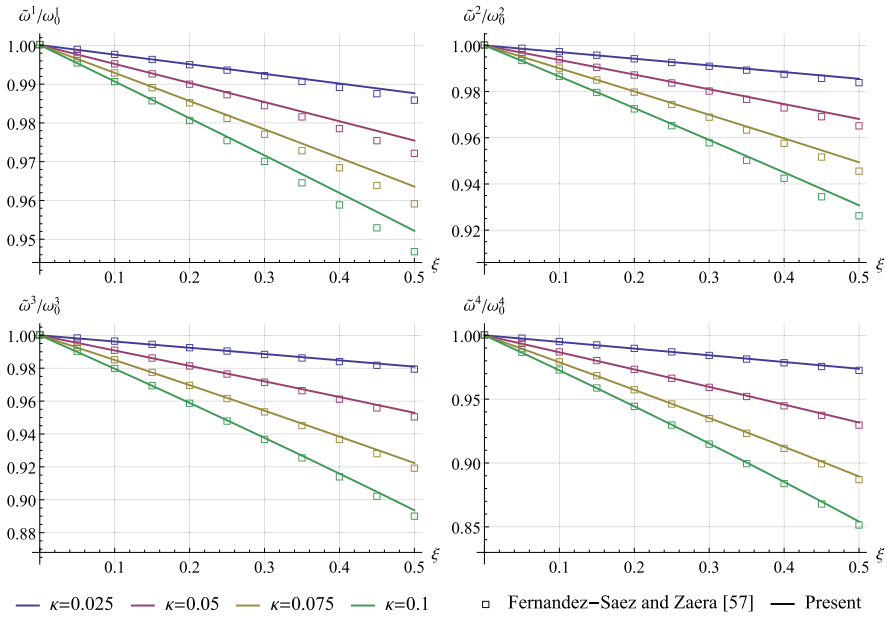

Fig. 2 Variation of the first-order ξ-expansions for the first four eigenvalues of a clamped-free beam

expressions provided herein may well pave the way to rapid optimization processes required for those aims.

In Fig. 1 and in the subsequent Fig. 2 we present some graphs showing the ratio of the first four ξ-incremental natural angular frequencies $\tilde{\omega}^k = \sqrt{\tilde{\lambda}^k}, k = 1, 2, 3, 4$, and the corresponding local natural angular frequencies $\omega_0^k = \sqrt{\lambda_0^k}, k = 1, 2, 3, 4$, of a simply supported and a clamped-free nano-beam, respectively. These results are compared with the corresponding ones found in [57]; it is immediate to check that the results coincide almost always, with slight discrepancies only for higher non-local volume fractions, more remarkable for the clamped beam. Such discrepancies always show a stiffening aspect of our procedure, which are to be expected because of the coarse approximation implied by the performed perturbation approach in terms of ξ. The higher discrepancies for the cantilever beam are justified by the fact that the curvature in such a beam is, the rest being fixed, more remarkable with respect to that of a simply supported beam: then, the possible stiffening aspect, linked to the hypothesis of purely flexibility, becomes more evident.

All curves start from the value 1 attained for $\xi = 0$, i.e., their values coincide with those of wholly local elastic beams when the non-local fraction vanishes, which was well expected.

All curves are monotonically decreasing with respect to ξ, which has a clear physically motivation: indeed, the introduction of a portion ξ behaving non-locally has a softening effect. This is due to the fact that, even though there are more 'springs' connecting each body-point to the others, the Gaussian-like non-local response to the same deformation has an integral effect which is below unity (it would be so on an infinite domain), and such a lower response is amplified by the volume fraction in the mixture, which is again lower than unity; all this is easily readable in Eqs. (5), (6).

In addition, the curves for greater values of the characteristic non-local length κ are always below those for lower κ; this also has a clear physical meaning, since the greater the radius of the long-range material interactions, the more the softening effect. To end with, it is also apparent, and physically motivated, that the ratios for simply supported beams attain higher values with respect to the corresponding ones for cantilever beams, always because the presence of two constraints limits the deformation of the former with respect to that of the latter, all the rest being fixed, and this implies a stiffer response.

6 Conclusions

We presented a perturbation procedure that lets truncated expansions of the natural angular vibration of purely flexible nano-beams, the behaviour of which is ruled by a mixture of linear elastic local and non-local mechanical response, be found without resorting to the much debated boundary constitutive conditions that are inherent to Eringen's model of non-local elasticity. We remark that, while the Euler-Bernoulli beam model and a constitutive law for a mixture of local and non-local linear elastic materials are well known in the literature, the original contribution of this work relies in this perturbation approach. Indeed, considering the non-local behaviour as a perturbation of the local one lets the non-local fraction be considered as a perturbation parameter; a formal series expansion provides a hierarchy of bulk equations completed by the usual boundary conditions of local elasticity. Thus, we reduce the nonlocal problem, which so many debates still originates, to a series of local problems where the nonlocal fraction, operating as a perturbation parameter, is kept into account by the forcing terms at the right-hand side of every step of the hierarchy. This resolution procedure is original, as far as we know, and detaches from previous resolution procedures of similar problems present in the literature; in addition, relying on well-known perturbation methods, it permits new ways of semi-analitical procedures in order to find agreeable solutions. Indeed, the search for non-trivial solutions asks for solvability conditions at successive steps of the hierarchy that let incremental natural angular frequencies of transverse vibration be found in closed form. It is not necessary to remark how important from the point of design and identification procedures the possibility to have closed-form solutions at ease is. The obtained results are compared with those of existing literature, showing good agreement and adherence to physical interpretation. Such results can be of interest in the modelling and parameter identification of nano-beams and in all the mechanical problems related.

Acknowledgements This work was initiated when U. Eroglu was Visiting Professor at the Dipartimento d'ingegneria strutturale e geotecnica of the University of Rome "La Sapienza", the support of which under the grant 'Professore visitatore 2020' CUP B82F20001090001 is gratefully acknowledged. G. Ruta acknowledges the financial support of the institutional grants RM11916B7ECCFCBF and RM12017294D1B7EF of the University "La Sapienza", Rome, Italy, and of the Italian national research grant PRIN 20177TTP3S-006 from Italian Ministry of University and Research.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix: Details and passages

In the following subsections we will present with more detail some of the passages omitted in the text for sake of space and readability.

Perturbation expansion

The governing equations for free vibration of Euler-Bernoulli nanobeams composed of the above described mixture of linear local and nonlocal elastic materials in terms of nondimensional quantities are obtained by replacing Eq. (9) into Eq. (8):

$$\begin{aligned} &\frac{dv}{dz}=-\Omega, \quad (1-\xi)\frac{d\Omega}{dz}+\xi K * \frac{d\Omega}{dz}=M, \\ &\frac{dT}{dz}=-\lambda v, \qquad\qquad \frac{dM}{dz}=T, \end{aligned} \tag{1}$$

Inserting the formal power series expansions of the field functions and of the eigenvalue up to the second order in the nonlocal fraction ξ we have

$$\begin{aligned} &\frac{dv_0}{dz}+\xi\frac{dv_1}{dz}+\frac{\xi^2}{2}\frac{dv_2}{dz}= \\ &\quad =-\Omega_0-\xi\Omega_1-\frac{\xi^2}{2}\Omega_2, \\ &\frac{d\Omega_0}{dz}+\xi\left(-\frac{d\Omega_0}{dz}+\frac{d\Omega_1}{dz}+K*\frac{d\Omega_0}{dz}\right)+ \\ &\quad +\xi^2\left(-\frac{d\Omega_1}{dz}+\frac{1}{2}\frac{d\Omega_2}{dz}+K*\frac{d\Omega_1}{dz}\right)= \\ &\quad =M_0+\xi M_1+\frac{\xi^2}{2}M_2, \\ &\frac{dT_0}{dz}+\xi\frac{dT_1}{dz}+\frac{\xi^2}{2}\frac{dT_2}{dz}=-\lambda_0 v_0-\xi\big(\lambda_0 v_1+ \\ &\quad +\lambda_1 v_0\big)-\frac{\xi^2}{2}\big(\lambda_0 v_2+\lambda_1 v_1+\lambda_2 v_0\big), \\ &\frac{dM_0}{dz}+\xi\frac{dM_1}{dz}+\frac{\xi^2}{2}\frac{dM_2}{dz}=T_0+\xi T_1+\frac{\xi^2}{2}T_2, \end{aligned} \tag{2}$$

By collecting like powers of ξ we obtain the expressions reported in Eqs. (13)–(15).

Fundamental matrix

For $z_0 = 0$, the components of the fundamental matrix $\mathbf{Y}(z, 0)$ are listed below.

$$
\begin{aligned}
Y_{11} &= \frac{1}{2}\left(\cos\lambda_0^{1/4}z + \cosh\lambda_0^{1/4}z\right)\\
Y_{12} &= -\frac{\sin\lambda_0^{1/4}z + \sinh\lambda_0^{1/4}z}{2\lambda_0^{1/4}}\\
Y_{13} &= \frac{\sin\lambda_0^{1/4}z - \sinh\lambda_0^{1/4}z}{2\lambda_0^{3/4}}\\
Y_{14} &= \frac{\cos\lambda_0^{1/4}z - \cosh\lambda_0^{1/4}z}{2\lambda_0^{1/2}}, \quad Y_{21} = \lambda_0 Y_{13},\\
Y_{22} &= Y_{11}, \quad Y_{23} = -Y_{14}, \quad Y_{24} = -Y_{12}\\
Y_{31} &= \lambda_0 Y_{12}, \quad Y_{32} = -\lambda_0 Y_{14}, \quad Y_{33} = Y_{11}\\
Y_{34} &= -\lambda_0 Y_{13}, \quad Y_{41} = \lambda_0 Y_{14}, \quad Y_{42} = -\lambda_0 Y_{13}\\
Y_{43} &= -Y_{12}, \quad Y_{44} = Y_{11}.
\end{aligned}
\tag{3}
$$

Orthogonality of the eigenmodes

By using the inner product defined in Eq. (20), the condition of orthogonality for the family of eigensolutions $\{M_0^j\}$ associated with the j-th eigenvalue λ_0^j is

$$
\left\langle DM_0^l, M_0^m \right\rangle = \int_0^1 \left[(M_0^l)^{IV} M_0^m - \lambda_0^l M_0^l M_0^m\right] dz = 0 \tag{4}
$$

This may be written, by the aid of variational principles,as

$$
\begin{aligned}
\left\langle DM_0^l, M_0^m \right\rangle = &\int_0^1 \left[(M_0^l)''(M_0^m)'' - \lambda_0^l M_0^l M_0^m\right] dz\\
&+\left[(M_0^l)''' M_0^m - (M_0^l)''(M_0^m)'\right]_0^1
\end{aligned}
\tag{5}
$$

In the previous equation, the boundary terms vanish due to Eq. (27); then, easy calculations show that

$$
\left\langle M_0^l, DM_0^m \right\rangle = \int_0^1 \left[(M_0^l)''(M_0^m)'' - \lambda_0^m M_0^l M_0^m\right] \tag{6}
$$

Then, by subtraction of the previous two equations,

$$
\begin{aligned}
&\left\langle DM_0^l, M_0^m \right\rangle - \left\langle M_0^l, DM_0^m \right\rangle =\\
&= (\lambda_0^m - \lambda_0^l)\int_0^1 M_0^l M_0^m dz = 0
\end{aligned}
\tag{7}
$$

By recalling Eq. (20) and accounting for the inequality $\lambda_0^m \neq \lambda_0^l \; \forall\, l \neq m$, the last equality of the previous equation leads to the searched orthogonality condition

$$
\left\langle M_0^l, M_0^m \right\rangle = 0 \tag{8}
$$

Fredholm Compatibility

Inserting the eigensolution expansion into the first-order equation for the l^{th} mode gives

$$
\begin{aligned}
&(M_0^l)^{IV} + \sum_{l\neq j} b_{lj}(M_0^j)^{IV} - \lambda_0^l M_0^l - \lambda_0^l \sum_{l\neq j} b_{lj} M_0^j =\\
&= (M_0^l)^{IV} - K * (M_0^l)^{IV} + \lambda_1 M_0^l
\end{aligned}
\tag{9}
$$

Inserting in Eq. (9) the zeroth-order field equation, Eq. (22)-1, we obtain

$$
\begin{aligned}
&\sum_{l\neq j} b_{lj}\lambda_0^j M_0^j - \lambda_0^l \sum_{l\neq j} b_{lj} M_0^j =\\
&= \lambda_0^l M_0^l - K * (\lambda_0^l M_0^l) + \lambda_1^l M_0^l
\end{aligned}
\tag{10}
$$

Multiplying both sides by M_0^m gives

$$
\begin{aligned}
&\sum_{l\neq j} b_{lj}\lambda_0^j M_0^j M_0^m - \lambda_0^l \sum_{l\neq j} b_{lj} M_0^j M_0^m =\\
&= \lambda_0^l M_0^l M_0^m - \lambda_0^l (K * M_0^l) M_0^m + \lambda_1^l M_0^l M_0^m
\end{aligned}
\tag{11}
$$

and integrating both sides over the domain provides, in terms of the inner product defined in Eq. (20),

$$
\begin{aligned}
&\sum_{l\neq j} b_{lj}\lambda_0^j \left\langle M_0^j, M_0^m \right\rangle - \lambda_0^l \sum_{l\neq j} b_{lj} \left\langle M_0^j, M_0^m \right\rangle =\\
&= \lambda_0^l \left\langle M_0^l, M_0^m \right\rangle - \lambda_0^l \left\langle (K * M_0^l), M_0^m \right\rangle\\
&+\lambda_1^l \left\langle M_0^l, M_0^m \right\rangle
\end{aligned}
\tag{12}
$$

When $l = m$ we have

$$
\begin{aligned}
&\sum_{m\neq j} b_{mj}\lambda_0^j \left\langle M_0^j, M_0^m \right\rangle - \lambda_0^m \sum_{m\neq j} b_{mj} \left\langle M_0^j, M_0^m \right\rangle =\\
&= \lambda_0^m \left\langle M_0^m, M_0^m \right\rangle - \lambda_0^m \left\langle (K * M_0^m), M_0^m \right\rangle +\\
&+\lambda_1^m \left\langle M_0^m, M_0^m \right\rangle
\end{aligned}
\tag{13}
$$

Exploiting the orthogonality of eigensolutions proved above, $\forall j \neq m$, $\left\langle M_0^m, M_0^j \right\rangle = 0$, Eq. (13) turns out to be

$$
\begin{aligned}
&\lambda_0^m \left\langle M_0^m, M_0^m \right\rangle - \lambda_0^m \left\langle (K * M_0^m), M_0^m \right\rangle +\\
&+\lambda_1^m \left\langle M_0^m, M_0^m \right\rangle = 0
\end{aligned}
\tag{14}
$$

which provides Eq. (37).

References

1. Capecchi D, Ruta G, Trovalusci P (2010) From classical to Voigt's molecular models in elasticity. Arch Hist Exact Sci 64:525–559
2. Capecchi D, Ruta G, Trovalusci P (2011) Voigt and Poincaré's mechanistic-energetic approaches to linear elasticity and suggestions for multiscale modelling. Arch Appl Mech 81:1573–1585
3. Capecchi D, Ruta G (2015) Strength of materials and theory of elasticity in 19th century Italy. Springer, Cham
4. Navier CL (1827) Mémoire sur les lois de l'équilibre et du mouvement des corps solides élastiques. Mémoires de l'Académie des Sci de l'Institut de Fr 7:375–393
5. Cauchy A (1828) Sur les équations qui expriment les conditions d'équilibre ou les lois du mouvement intérieur d'un corps solide, élastique ou nonélastique. Exerc de Mathématiques 3:160–187
6. Poisson CLMH (1828) Mémoire sur l'équilibre et le mouvement des corps élastiques. Mémoires de l'Académie des Sci de l'Institut de Fr 8(2):357–570
7. Green G (1842) On the laws of the reflexion and refraction of light at the common surface of two non-crystallized media. Trans Camb Philos Soc 7:1–24
8. Dell'Isola F, Della Corte A, Esposito R, Russo L (2016) *Generalized continua as models for classical and advanced materials*. In: Some cases of unrecognized transmission of scientific knowledge: from antiquity to gabrio piola's peridynamics and generalized continuum theories. Advanced Structured Materials. Springer
9. Cosserat EMP, Cosserat F (1909) Théorie des corps déformables. A. Hermann, Paris
10. Voigt W (1900) L'état actuel de nos connaissances sur l'élasticité des cristaux. Rapports présentés au Congrés international de Physique. Gauthier-Villars, Paris
11. Poincaré H (1892) Leçons sur la théorie de l'élasticité. Carré, Paris
12. Wieghardt K (1922) Über den balken auf nachgiebiger unterlage. Zeitschrift für angewandte mathematik und mechanik - ZAMM 2(3):165–184 (**in German**)
13. Love AEH (1906) *A treatise on the mathematical theory of elasticity*. At the University Press, Cambridge, 2. edition
14. Ridha AA, Raad MF, Nadhim MF (2019) Nano composite - introduction & application. Noor Publishing/Omniscriptum, Chisinau, Moldova
15. Trovalusci P (2015) *Discrete to Scale-Dependent Continua for Complex Materials: A Generalized Voigt Approach Using the Virtual Power Equivalence in Materials with Internal Structure: Multiscale and Multifield Modeling and Simulation*. Springer Tracts in Mechanical Engineering. Springer International Publishing
16. Rapaport DC (2004) The art of molecular dynamics simulation. Cambridge University Press
17. Reccia E, Leonetti L, Trovalusci P, Cecchi A (2018) A multiscale/multidomain model for the failure analysis of masonry walls: a validation with a combined fem/dem approach. Int J Multiscale Comput Eng 16:325–343
18. Kunin IA (1982) Elastic media with microstructure I: one-dimensional models. Springer
19. Maugin GA (1979) Nonlocal theories or gradient-type theories: a matter of convenience. Arch Mech 31(1):15–26
20. Eringen AC (1999) Microcontinuum field theory. Springer
21. Eringen AC, Suhubi ES (1964) Nonlinear theory of simple micro-elastic solids. Int J Eng Sci 2(2):189–203
22. Mindlin RD, Tiersten HF (1962) Effects of couple-stresses in linear elasticity. Arch Ration Mech Anal 11:415–448
23. Kröner E (1967) Elasticity theory of materials with long range cohesive forces. Int J Solids Struct 3(5):731–742
24. Eringen AC, Edelen DGB (1972) On nonlocal elasticity. Int Jo Eng Sci 10:233–248
25. Eringen AC (2002) Nonlocal continuum field theories, 1st edn. Springer
26. Kunin IA (1983) Elastic media with microstructure II: three-dimensional models. Springer, Cham
27. Trovalusci P, Capecchi D, Ruta G (2008) Genesis of the multiscale approach for materials with microstructure. Arch Appl Mech 79(11):981
28. Maugin GA (2010) Generalized continuum mechanics: What do we mean by that? Mechanics of generalized continua: one hundred years after the cosserats, advances in mechanics and mathematics 21. Springer, New York
29. Patrizia Trovalusci (2014) Molecular approaches for multifield continua: origins and current developments. Springer Vienna, Vienna
30. Mohamed Shaat, Esmaeal Ghavanloo, Ahmad Fazelzadeh V (2020) Review on nonlocal continuum mechanics: physics, material applicability, and mathematics. Mech Mater 150:103587
31. Polizzotto C (2016) A note on the higher order strain and stress tensors within deformation gradient elasticity theories: physical interpretations and comparisons. Int J Solids Struct 90:116–121
32. Meral Tuna, Patrizia Trovalusci (2020) Scale dependent continuum approaches for discontinuous assemblies:"explicit" and "implicit" non-local models. Mech Res Commun 103:103461
33. Tuna M, Leonetti L, Trovalusci P, Kirca M (2020) "Explicit" and "implicit" non-local continuous descriptions for a plate with circular inclusion in tension. Meccanica 55:927–944
34. Eringen AC (1966) A unified theory of thermomechanical materials. Int J Eng Sci 4(2):179–202
35. Eringen AC (1977) Screw dislocation in non-local elasticity. J Phys D Appl Phys 10(5):671–678
36. Aydogdu M (2009) A general nonlocal beam theory: its application to nanobeam bending, buckling and vibration. Physica E 41(9):1651–1655
37. Roque CMC, Ferreira AJM, Reddy JN (2011) Analysis of Timoshenko nanobeams with a nonlocal formulation and meshless method. Int J Eng Sci 49(9):976–984
38. Uymaz B (2013) Forced vibration analysis of functionally graded beams using nonlocal elasticity. Compos Struct 105:227–239
39. Aranda-Ruiz J, Loya J, Fernández-Sáez J (2012) Bending vibrations of rotating nonuniform nanocantilevers using the Eringen nonlocal elasticity theory. Compos Struct 94(9):2990–3001
40. Mao JJ, Lu HM, Zhang W, Lai SK (2020) Vibrations of graphene nanoplatelet reinforced functionally gradient

piezoelectric composite microplate based on nonlocal theory. Compos Struct 236:111813
41. Maysam Naghinejad and Hamid Reza Ovesy (2018) Free vibration characteristics of nanoscaled beams based on nonlocal integral elasticity theory. J Vib Control 24(17):3974–3988
42. Pei Z, Hai Q (2022) Free vibration analysis of Euler-Bernoulli curved beams using two-phase nonlocal integral models. J Vib Control 28:2861–2878
43. Pisano AA, Fuschi P, Polizzotto C (2021) Integral and differential approaches to Eringen's nonlocal elasticity models accounting for boundary effects with applications to beams in bending. ZAMM - J Appl Math Mech/ Zeitschrift für Angewandte Mathematik und Mechanik 101(8):e202000152
44. Challamel N, Wang CM (2008) The small length scale effect for a non-local cantilever beam: a paradox solved. Nanotechnology 19(34):345703
45. Fernández-Sáez J, Zaera R, Loya JA, Reddy JN (2016) Bending of Euler-Bernoulli beams using Eringen's integral formulation: a paradox resolved. Int J Eng Sci 99:107–116
46. Tuna M, Kirca M (2016) Exact solution of Eringen's nonlocal integral model for bending of Euler-Bernoulli and Timoshenko beams. Int J Eng Sci 105:80–92
47. Tuna M, Kirca M (2016) Exact solution of Eringen's nonlocal integral model for vibration and buckling of Euler-Bernoulli beam. Int J Eng Sci 107:54–67
48. Romano G, Barretta R (2016) Comment on the paper "Exact solution of Eringen's nonlocal integral model for bending of Euler-Bernoulli and Timoshenko beams" by Meral Tuna and Mesut Kirca. Int J Eng Sci 109:240–242
49. Romano G, Barretta R, Diaco M, de Sciarra FM (2017) Constitutive boundary conditions and paradoxes in nonlocal elastic nanobeams. Int J Mech Sci 121:151–156
50. Tuna M, Kirca M (2017) Respond to the comment letter by Romano and Barretta on the paper exact solution of Eringen's nonlocal integral model for bending of Euler-Bernoulli and Timoshenko beams. Int J Eng Sci 116:141–144
51. Eringen AC (1972) Linear theory of nonlocal elasticity and dispersion of plane waves. Int J Eng Sci 10:425–435
52. Benvenuti E, Simone A (2013) One-dimensional nonlocal and gradient elasticity: closed-form solution and size effect. Mech Res Commun 48:46–51
53. Wang YB, Zhu XW, Dai HH (2016) Exact solutions for the static bending of Euler-Bernoulli beams using Eringen's two-phase local/nonlocal model. AIP Adv 6(8):085114
54. Wang Y, Huang K, Zhu X, Lou Z (2019) Exact solutions for the bending of Timoshenko beams using Eringen's two-phase nonlocal model. Math Mech Solids 24(3):559–572
55. Tuna M, Kirca M, Trovalusci P (2019) Deformation of atomic models and their equivalent continuum counterparts using Eringen's two-phase local/nonlocal model. Mech Res Commun 97:26–32
56. Eroğlu U (2020) Perturbation approach to Eringen's local/ non-local constitutive equation with applications to 1-d structures. Meccanica 55:1119–1134
57. Fernández-Sáez J, Zaera R (2017) Vibrations of Bernoulli-Euler beams using the two-phase nonlocal elasticity theory. Int J Eng Sci 119:232–248
58. Fakher M, Hosseini-Hashemi S (2020) Vibration of two-phase local/nonlocal Timoshenko nanobeams with an efficient shear-locking-free finite-element model and exact solution. *Engineering with Computers*, pages 232–248
59. Fakher M, Hosseini-Hashemi S (2021) Nonlinear vibration analysis of two-phase local/nonlocal nanobeams with size-dependent nonlinearity by using Galerkin method. J Vib Control 27(3–4):378–391
60. Khaniki HB (2018) Vibration analysis of rotating nanobeam systems using Eringen's two-phase local/nonlocal model. Physica E 99:310–319
61. Zaera R, Serrano Ó, Fernández-Sáez J (2019) On the consistency of the nonlocal strain gradient elasticity. Int J Eng Sci 138:65–81
62. Barretta R, Marotti de Sciarra F (2019) Variational nonlocal gradient elasticity for nano-beams. Int J Eng Sci 143:73–91
63. Zaera R, Serrano Ó, Fernández-Sáez J (2020) Non-standard and constitutive boundary conditions in nonlocal strain gradient elasticity. Meccanica 55:469–479
64. Ceballes S, Larkin K, Rojas E, Ghaffari SS, Abdelkefi A (2022) Nonlocal elasticity and boundary condition paradoxes: a review. J Nanopart Res 23:66
65. Pignataro M, Rizzi N, Luongo A (1991) Stability, bifurcation and postcritical behaviour of elastic structures. Elsevier
66. Ruta G, Pignataro M, Rizzi N (2006) A direct one-dimensional beam model for the flexural-torsional buckling of thin-walled beams. J Mech Mater Struct 1(8):1479–1496
67. Pease MC (1965) Methods of matrix algebra. Academic Press, New York
68. Eroglu U, Ruta G, Tufekci E (2019) Natural frequencies of parabolic arches with a single crack on opposite cross-section sides. J Vib Control 25(7):1313–1325