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Research Article

Sixth-grade students' pattern generalization approaches

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This study investigates sixth-grade Turkish students' pattern-generalization approaches among arithmetical generalization, algebraic generalization, and naïve induction. A qualitative case study design was employed. The data was collected from four sixth-grade students through the Pattern Questionnaire (PQ) and individual interviews based on the questionnaire. The findings revealed that all students generalized near terms using arithmetical generalization as the first step and then they mostly looked for a general rule through memorized procedures by skipping far term generalization. When they found the general rule, far terms were calculated by rote. In other words, students did not generalize the pattern to far terms using an algebraic generalization. The current study's findings would give valuable information to the mathematics educators regarding the necessity of avoiding creating a procedural instructional environment by focusing on the rote procedure of finding the general rule of a pattern. These findings would also expand the horizons of curriculum developers regarding the importance of objectives about both near terms and far term generalization by progressing from arithmetical generalization to algebraic generalization.

Keywords: Algebra; Pattern; Generalization; Arithmetic; Algebraic; Symbolic

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1. Introduction

Algebra learning is considered the key to the development of some basic mathematical skills such as understanding equality, understanding patterns, relational thinking, and problem solving. It is strongly recommended to involve algebra instruction as a part of the middle school mathematics curriculum (Carraher et al., 2006; National Council Teachers of Mathematics [NCTM], 2010). Elementary mathematics curricula of most countries, including Türkiye, acquaint students with algebra in the middle grades (6^{th}-8^{th}-grade level) after five years of arithmetic-based instruction at the primary grades. Nevertheless, literature shows a serious obstacle for young students to develop algebraic reasoning skills because of the dominance of arithmetic in primary grades (Kamol & Har, 2010; Warren, 2003). State differently, beginning algebra students experience severe difficulty during the transition from arithmetic to algebra (Knuth et al., 2005; Sfard, 1995).

To ensure the transition from arithmetic to algebra, the literature recommends pattern-generalizing tasks (Dörfler, 2008). Pattern generalization tasks enable great opportunities for

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students to gain experience in both arithmetic and algebra fields. It is because a pattern can be generalized both arithmetically and algebraically. To conduct the generalization arithmetically, an arithmetic/additive relationship between terms is explored; on the other hand, to conduct the generalization algebraically, it is necessary to explore the algebraic relationship between terms and term numbers. The transition from former to latter could get easier as students deal with pattern generalization activities. Yet, it is extremely important to make sure that the transition process is smooth and natural (Carpenter et al., 2003). Students will likely not build a bridge between two fields when there is a sharp jump from arithmetic generalization to algebraic generalization.

To provide a smooth transition from arithmetic to algebraic generalization, literature recommended focusing on the generalization of near and far terms of the pattern before any/n^{th} term (NCTM, 1997; Radford, 2008). In an ordered pattern, it is natural and easy to use arithmetic/additive relationships when it is asked to generalize the pattern to near terms such as the fifth or sixth term. Yet, generalizing the pattern to far terms such as 20^{th}, 50^{th}, or 100^{th} terms would not be as practical as near terms. It is necessary to move from the additive structure of consecutive terms to the algebraic structure of each term based on term number. Thus, the transition from near term generalization to far term generalization would naturally lead students from arithmetic thinking to algebraic thinking (Ontario Ministry of Education, 2013). In this way, pattern generalization tasks enable researchers to observe students' arithmetic and algebraic generalization processes.

To be able to focus on students' transition from arithmetic to algebraic generalization process, it is important to observe and report the whole generalization process of students. When the literature is reviewed, it is seen that while some studies (Akkan & Çakıroğlu, 2012; Firdaus et al., 2019; Lin & Yang, 2004) resulted in one generalization method while describing students' pattern generalization, some of the studies (Amit & Neria, 2008; Barbosa, 2011; Orton & Orton, 1999; Stacey, 1989) reported that students are more likely to use more than one generalization method since near and far generalizations might necessitate different processes. The latter aspect provides evidence that pattern generalization cannot be restricted to one method since it represents a developmental process (Radford, 1996). While students change the generalization method as they develop from near term generalization to far term generalization and to find the general term, it becomes possible to observe developmental generalization processes and also detect the possible gaps during the transition from arithmetic generalization to algebraic generalization.

Algebraic generalization is a process as well as pattern generalization (Radford, 2010a). Yet, some studies do not represent algebraic generalization developmentally. For example, the method with algebraic nature was called *linear strategy* (Stacey, 1989), *functional/conceptual/global strategy* (Amit & Neria, 2008), *explicit/linear strategy* (Orton & Orton, 1999), and *correspondence relationship* (Somasundram, 2019). These methods often carry similar meanings despite being called differently by researchers. These researchers represent algebraic generalization as a one-step method, yet this view might represent a restricted approach. On the other hand, in parallel with the process approach, some recent studies began to view algebraic generalization as a process (Aké et al., 2013; Godino et al., 2015; Maudy et al., 2018; Radford, 2010a), so they defined algebraic generalization methods not as limited with one method but as a combination of developmental layers. For example, Aké et al. (2013) defined four levels of algebraic generalization, and Radford (2010a) divided algebraic generalization into three progressive steps. This developmental approach enables especially beginning algebra students to develop algebraic thinking skills step by step. Therefore, avoiding a sharp transition from arithmetic generalization to algebraic generalization would become more possible.

All in all, it can be said that pattern generalizing tasks have great importance for beginning algebra students during the transition from arithmetic to algebra since they enable them to experience and relate both arithmetic and algebraic generalization processes. Still, it is necessary to avoid a sharp transition from arithmetic to algebraic generalization. To provide this transition as smoothly as possible, the literature recommends seeing algebraic generalization as a process and moving from the generalization of given terms to the generalization of not given near and then far terms. In this study, students' generalization process was investigated in a sequence of near term generalization, far term generalization, and reaching the general term through a developmental approach as recommended by the literature. In this way, it would be possible to explore students' own generalization paths regarding near term generalization, far term generalization, and reaching the general term. Thus, it would be possible to detect students' gaps between arithmetic generalization and algebraic generalization. Based on the literature review, we focused on the sixth-grade Turkish students' generalization of patterns in algebra.

1.1. Theoretical Framework

As explained above, pattern generalization tasks carry a progressive nature from arithmetic generalization to algebraic generalization. Thus, while selecting the theoretical perspective of the present study, this progressive nature was considered as the main concern. In the present study, the pattern generalization approach by Radford was adopted as the theoretical perspective. Radford (2006, 2008, 2010a) has studied the field of algebraic generalization. After long years of study, he observed a gap between students' starting algebraic thinking and their capability of using symbolic algebra (Radford, 2010b). Through his theory, he called this gap a "zone of the emergence of algebraic thinking" (Radford, 2010b, p.36). According to Radford, algebraic generalization has progressive layers (Radford et al., 2006). Students should pass from each layer to ensure the full transition from arithmetical generalization to algebraic generalization (Radford, 2003). These layers are factual generalization, contextual generalization, and symbolic generalization.

Arithmetical generalization refers to the generalization based on the recursive relationship between consecutive terms of the pattern. For example, if a student notices that 2 more circles are added in each step of the pattern in Figure 1, this is an example of arithmetical generalization.

Figure 1
The pattern example from Radford et al. (2006, p. 395)

After arithmetical generalization, the first layer of algebraic generalization, i.e., factual generalization, progresses. Through factual generalization, students realize a "factual" generality, which allows for building the algebraic relationship between term numbers and terms (Radford, 2003, p.46). This factual generality enables students to find the numerical value of near and far terms. When a student explored that the top row of the first step has 2 circles, which is 1 more than step number 1, and the bottom row has 3 circles, which is 2 more than step number 1, it could be a starting point for factual generalization for the pattern in Figure 1. Then, s/he might search it in the second or third steps and notice that the second and third steps also have the same structure: the top row of the second step has 3 circles, which is 1 more than step number 2, and the bottom row has 4 circles, which is 2 more than the step number 2. This exploration of the student is a factual generalization. For example, it enables us to find the number of circles in the 25^{th} step as 26 circles in the top row and 27 circles in the bottom row.

After factual generalization, students tend to find unspecific terms. They move to express the general rule to find any term of the pattern. If they use natural language, it is called contextual generalization (Radford, 2003). For instance, the expression of the general rule of the pattern in Figure 1 as 'The top row always has 1 more circle than the step number and the bottom row has 2 more circles than the step number' is an example of contextual generalization. On the other hand, if they use formal symbolic language, it is called symbolic generalization (Maudy et al., 2018; Radford, 2010a). Traditionally, the formal symbolic language of algebra includes alphanumeric letters such as x, y, z, n, etc. These letters are used in replacement of unspecified term numbers.

For example, the general rule of the pattern in Figure 1 can be expressed as '(n+1)+(n+2)' through symbolic generalization.

Apart from arithmetical and algebraic generalization, Radford also defined naïve induction. Naïve induction does not carry the nature of generalization (Radford, 2010a). It includes trial and error without understanding the mathematical structure of terms. Through naïve induction, students just try some calculations to find a rule that fits the pattern. For example, a student might try to multiply step number 2 with 3 and add 1 in the pattern of Figure 1. Yet, it does not work for the first and third steps. Then, s/he might try to multiply it by 2 and add 3. Luckily, this trial works for other steps. Then, students might write this trial rule as a general rule. Still, this process does not represent the nature of generalization.

In sum, Radford's pattern generalization approach carries a progressive nature and enables distinguishing trial and error from the arithmetic and algebraic generalization processes. In the present research, students' generalization approaches were analyzed within the scope of Radford's generalization layers.

1.2. The Rationale for the Study

Despite the considerable gains of pattern generalization, some problematic areas come into existence, which prevent the conceptual understanding of pattern generalization in educational environments. The first problematic area identified in the literature is the procedure-focused instruction of pattern generalization in mathematics lessons. Literature showed that the instruction of pattern generalization is conducted in a procedural way more than in a conceptual way in traditional mathematics lessons (Lannin et al., 2006). This procedure generally consists of finding the common difference and multiplying the common difference with the n^{th} term, where n is the term number, and replacing n with any specific term number in order to decide which number to add or subtract (Spangenberg & Pithmajor, 2020). As an example, Girit and Akyüz (2016) reported that students "get used to multiply something and add something for getting a rule" (p. 261). By memorizing the procedure, students could apply this technique, generate the rule of the pattern, and answer the questions, yet they cannot develop a conceptual understanding of pattern generalization in this way. To understand pattern generalization conceptually, literature recommended progressing from exploring arithmetic relationships to algebraic relationships. First, due to the easiness of noticing arithmetic differences, it would be normal for beginning algebra students to see the pattern as an increasing/decreasing number sequence with a fixed difference. Then, asking some leading questions could transform their understanding from the arithmetic stage to the more algebraic stage. For example, asking to reach some near terms such as 5^{th}, 6^{th}, 10^{th} and then some further terms such as 20^{th}, $30^{th,}$ and eventually far terms such as 100^{th}, 1000^{th} could make students feel the impracticality of arithmetic increase/decrease and lead them to explore algebraic relations within each term and term number. In this way, students explore patterns in a progressive conceptual way instead of memorizing the procedures.

The other problematic area of pattern generalization is the obsession with identifying students' algebraic generalization capabilities only with symbol use (Radford, 2010b). As literature showed, in order to reach algebraic generalization, students should naturally go through a progressive process (Aké et al., 2013; Garcia-Cruz & Martinón, 1998; Godino et al., 2015; Maudy et al., 2018; Radford, 2010a). This progressive process should include the development from pre-symbolic generalizations to symbolic generalizations (Rivera, 2013). Yet, lately, most of mathematics curricula, including Turkish mathematics curricula (Ministry of National Education [MoNE], 2018), expect beginning algebra students to manage symbolic generalization as a first step. Nonetheless, this approach prevents students from comprehending the progressive nature of pattern generalization (Lannin et al., 2006). According to Radford's theory of algebraic generalization, the absence of algebraic symbols does not show the absence of algebraic thinking. In addition, it restricts mathematics educators from viewing the generalization of students as tied up with the correct usage of symbolic generalization rather than as a developmental process.

As the literature showed, rote procedures of finding the general rule and persistence of symbolic generalization have become the center of mathematics instruction (Girit & Akyüz, 2016; Lannin et al., 2006; Maudy et al., 2018). Due to these problematic areas, students have difficulties in understanding algebraic generalization conceptually. According to Lee and Wheeler (1987), the difficulty for students is not "seeing the pattern" but "seeing an algebraically useful pattern" (p.95). Many studies reported students experiencing difficulty in understanding the pattern algebraically during pattern generalization (Becker & Rivera, 2005, 2006; Lee & Wheeler, 1987; Lin & Yang, 2004). For example, Becker and Rivera (2005) reported only five students out of 22 ninth-grade students generalized a pattern algebraically. Similarly, Maudy et al.'s (2018) study showed that students were not able to build a conceptual understanding of the algebraic nature of generalization. Thus, it is of great importance to analyze students' pattern generalization as a whole process from two aspects. First, it could enable researchers to follow the flow of each student's changing methods during generalization based on the near term, far term, and general term. Then, it could clarify the point of a sharp jump from arithmetical generalization to the rote procedures of symbolic generalization. Second, it could detect students' general tendency during passing from arithmetical generalization to the developmental layers of algebraic generalization.

All in all, as indicated in the literature, the problem statement is students' tendency to apply arithmetic generalization at the beginning and then jump either to rote-memorized procedures of finding the general rule or to meaningless trial and error by skipping progressive algebraic generalization layers. Yet, it is not clear at what stage students skip algebraic generalization layers and prefer rote memorized procedures. Also, there are not enough studies conducted with students in Türkiye in this sense. Therefore, this study aims to enlighten beginning algebra students' generalization processes in their entirety to determine their generalization approaches while generalizing a pattern to near terms, far terms, and any term. Based on these concerns, the research question of the current study is as follows:

- What is the nature of sixth grade students' generalization approaches in terms of arithmetical generalization, layers of algebraic generalization, and naïve induction?

2. Method

In the current study, a qualitative case study design was employed. The reason for selecting a case study design is that it enables the researcher to explore an issue deeply through cases, and it could involve data from multiple sources (Creswell, 2013). In this study, researchers aimed to explore students' pattern generalization approaches through cases by using written work and interviews. The data was collected from four sixth-grade public school students in Türkiye.

2.1. Pattern-generalization in Turkish Middle School Mathematics Curriculum

In the Turkish middle school mathematics curriculum, fifth-grade students are expected to expand an arithmetic pattern to some near terms, such as the fifth or sixth term, by adding the fixed difference onto the previous term (MoNE, 2018). Students are expected to express the rule of arithmetic patterns with the letter and to calculate the asked steps of the patterns whose rule is expressed with a letter in sixth-grade curriculum materials (MoNE, 2018).

Therefore, we focus on the sixth-grade students' mathematical activities when algebraic symbols were first introduced in the official curriculum materials.

2.2. Participants

The participants of the study were selected from a public middle school purposefully. There were two selection criteria used, the first of which is students' enthusiasm for mathematics lessons. The reason behind this criterion is the researchers' common opinion that mathematically enthusiastic students would be more interested in answering the questions in the Pattern Questionnaire and more likely to provide rich information. The second criterion was their willingness to talk about their thinking. In this study, data were gathered through individual interviews conducted one participant at a time, so the participants' willingness to talk was essential. All the students participated voluntarily in the current study. The participants of the study were selected from a public middle school. There were some commonalities among these four students. Firstly, all four students' age ranged from 11 to 12. Their socioeconomic statuses were medium, where monthly income was a bit more than the established subsistence level. P1, P2, P3, and P4 were defined with moderate academic performance by their mathematics teacher. They all were in the same class. All four students were talkative and enthusiastic about the mathematics lesson. They were given pseudonyms and labeled as P1, P2, P3, and P4.

To ensure ethical issues, official permissions were taken from the University Ethics Committee and Ministry of Education as the first step. Then, the school administration was informed about the study and the participants of the study were selected among voluntary students. Each participant was asked to sign the Informed Consent Form and Parental Approval Form. The confidentiality of identities was ensured during the study.

2.3. Data Collection

There were two sources of data: participants' written work on the Pattern Questionnaire and task-based interviews conducted during the Pattern Questionnaire. The Pattern Questionnaire included three open-ended pattern generalization tasks, all of which were adapted from the literature. Due to the adaptation of each task from the literature, the validity and reliability of the tasks were ensured in the past related studies. The purpose of the Pattern Questionnaire was to gather information about the pattern generalization process of participants. As recommended in the literature, pattern generalization objectives specifically included generalizing the pattern to the next, near, and far terms in order and expressing a general rule (Moss et al., 2006; Radford, 2000). Therefore, each question included four sub-questions related to generalizing the pattern to the (i) next, (ii) near, (iii) far terms, and (iv) expressing the general rule.

The other data source was task-based interviews (Goldin, 2000). The first author interviewed one participant at a time while the participant was answering the questions in the Pattern Questionnaire. For instance, as a student was generalizing the pattern and answering the sub-questions in the Pattern Questionnaire, the first author asked about how s/he generalized the pattern to near or far terms or asked about more detail of the solution. For example, while a student was generalizing a pattern in a question, the first author asked questions such as 'Could you explain your work here?', 'Could you explain what you meant in detail?' or 'How did you find the number of circles in the twentieth step?' The duration of the interview was almost one hour. The confidentiality of the identities and voluntary participation were reminded before starting each interview session. The sessions were audio recorded.

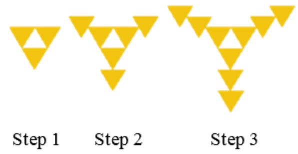
2.4. The Tasks in the Pattern Questionnaire

The first task was adapted from the study of Van de Walle et al. (2007). Originally (see Figure 2), it was asked to fill in the blanks in the given table for the first, second, third, fourth, fifth, tenth, and twentieth steps. In addition, it was asked to write a general rule with words and/or symbols. To not limit students in a specific direction, the table part of the question was removed, and the general rule was asked without directing students to use words or symbols. Lastly, triangles were replaced with circles since it is easy to draw circles.

The second task was adapted from the study of Mason et al. (2005). In the original version (see Figure 3), it was asked only to find a rule for the n^{th} picture. Near or far terms were not requested. In the adapted version, new items, which ask for near and far steps, such as the fourth, fifth, tenth, and fiftieth steps, were added to see students' generalization processes of near and far terms by Radford's generalization approach. Additionally, instead of 'n^{th} picture,' 'the general rule' expression was used not to lead students to use 'n.'

Figure 2

(Upper) The original version of question 1 (Van de Walle et al., 2007, p. 269); (Lower) the adapted version of question 1



- Complete a table that shows number of triangles for each step.

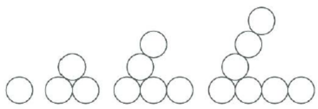
Step number	1	2	3	4	5…	10	20
Number of triangles							

- How many triangles are needed for step 10? Step 20? Step 100? Explain your reasoning.
- Write a rule (in words and/or symbols) that gives the total number of pieces to build any step number (n).

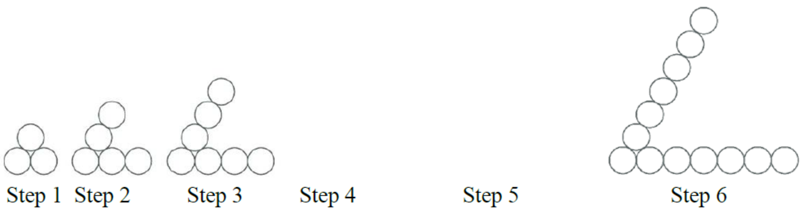
1. At the above pattern with circles,
a. How many circles does the fourth step have?
b. How many circles does the tenth step have?
c. How many circles does the fiftieth step have?
d. How can you express the general rule of the pattern?

Figure 3

(Upper) The original version of question 2 (Mason et al., 2005, p. 117); (Lower) the adapted version of question 2



For each of the picture sequences, decide on a rule that generates these and subsequent pictures in the sequence. How many objects (circles, squares) are needed to make the nth picture?

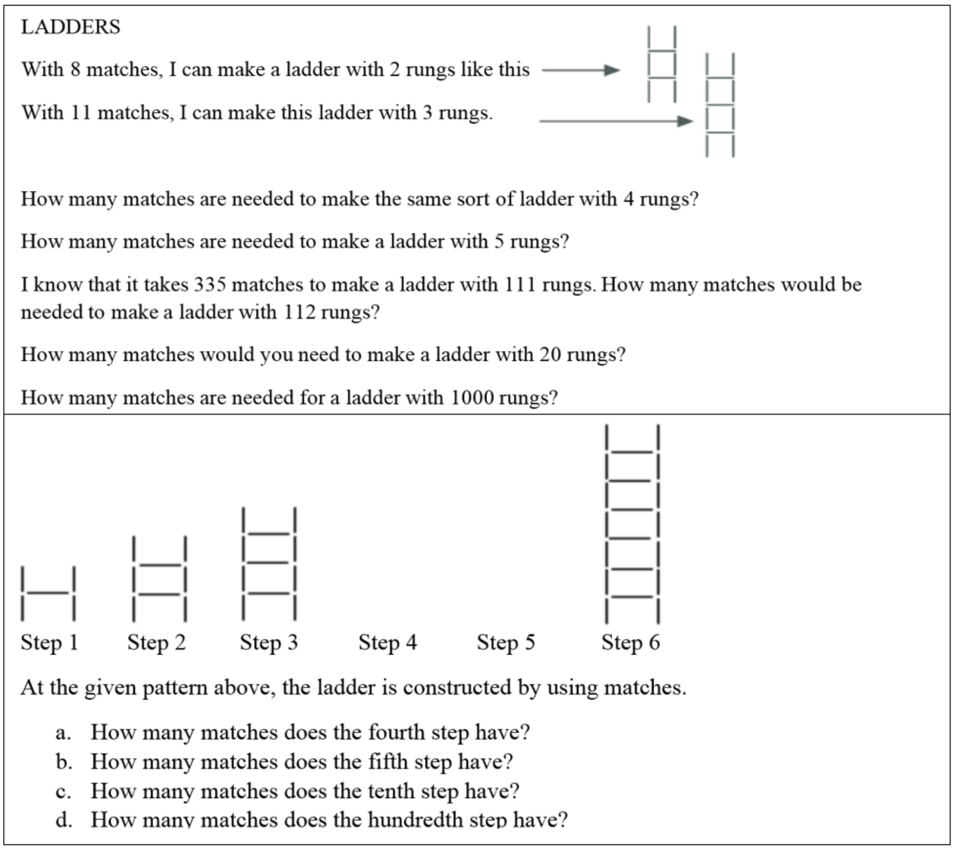


At the above pattern with circles,

a. How many circles does the fourth step have?
b. How many circles does the fourth step have?
c. How many circles does the tenth step have?
d. How many circles does the fiftieth step have?
e. How can you express the general rule of the pattern?

The third task was adapted from the study of Stacey (1989). In Stacey's study (1989), the pattern has presented the pattern with two visuals rather than a sequence of visuals. In addition, she used the expression 'the number of rungs' instead of 'step number'. Lastly, no question was asked about the general rule of the pattern, as seen in Figure 4. To use the question in the current study, the pattern was represented as growing steps. In addition, the phrase ‘step number’ was preferred instead of the phrase ‘the number of rungs’. Lastly, an item that asks for a general rule of the pattern was added to see how students generalize a pattern to any term according to the purpose of the study.

Figure 4
(Upper) The original version of question 3 (Stacey, 1989, p. 148); (Lower) the adapted version of question 3

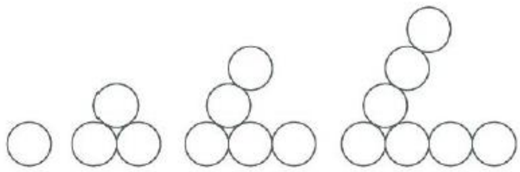


2.5. Data Analysis

During data analysis, all the audiotaped interviews were transcribed to prepare for descriptive analysis. Researchers read all the transcripts repeatedly and the first coding was managed based on predetermined codes (arithmetical generalization, algebraic generalization [with layers of factual generalization, contextual generalization, and symbolic generalization], and naïve induction, as obtained from Radford, 2000). In addition to these predetermined codes, it was also observed that at the end of the generalization processes, some students verified their rules by replacing the unknown quantity with some known numbers. This part of students’ answers was coded as verification. Lastly, as expressed in detail above, sub-questions of each question asked for the generalization of some near and far steps of the pattern such as fourth, fifth, tenth, hundredth, etc. If students skipped these generalization steps and calculated them by applying the general rule after finding a general rule, this part of their answers was coded as rote calculation. Secondly, the transcribed data were coded by another coder, who is also a mathematics teacher, based on a given coding schema. After coding the transcript by the first researcher and an independent coder, the results were compared. Lastly, the establishment of the last version of coded transcripts on codes and categories were provided.

As a result of data analysis, three categories emerged. The first category included (i) *generalizing near terms with Arithmetic Generalization*, (ii) *expressing the general rule with Contextual Generalization or Symbolic Generalization*, (iii) *Verification of the general rule*, and (iv) *Rote Calculation*. The second category included (i) *generalizing near terms with Arithmetic Generalization*, (ii) *finding a general rule* *with Naïve Induction*, and (iii) *rote calculation*. The third category included (i) *generalizing near terms with both Arithmetic Generalization and Factual Generalization*, (ii) *expressing the general rule with Contextual Generalization and/or Symbolic Generalization*, and (iii) *Rote Calculation (if needed)*. These categories were called Generalization Approaches in the rest of the study. Below are some examples of the codings based on the given pattern in Figure 5.

Figure 5
A pattern example from Mason et al. (2005, p.137)



If students start the generalizing process by adding the constant difference between consecutive terms to the previous term in order to find the near terms, it was coded as '*generalizing near terms with Arithmetical Generalization*'. For instance, 'This pattern increases by 2 at each step. So, I should add 2 to the fourth term in order to find the fifth term' that can be categorized under this code.

If students generalize the pattern to near terms both with Arithmetical Generalization and Factual Generalization, it was called *'generalizing near terms with both Arithmetic Generalization and Factual Generalization'*. An example for this category is as follows: 'This pattern increases by 2 at each step. So, I should add 2 to the fourth term in order to find the fifth term. The fourth term is 7. Then, the fifth term is 9. In the fourth term, the bottom line has 4 circles. It is the same with step number 4. The upper line has 3 circles. It is 1 less than step number 4. Then, in the fifth term, the bottom line will have 5 circles, and the upper line will have 4 circles. 4 plus 5 is 9. I had already found 9 for the fifth step.'

If students reach the general term with Contextual Generalization or Symbolic Generalization, it was coded as '*expressing the general rule with Contextual Generalization and/or Symbolic Generalization*'. Some examples for this category could be as follows: 'I should multiply the step number with 2, because the common difference is 2. Then, I should subtract 1. This is the rule. Then, it is 2n-1.' or 'The number of circles in the bottom line is the same as the step number. The number of circles in the upper line is one less than the step number. Then, the general rule is step number plus step number minus one. In other words, n+n-1.'

If students reach the general term with meaningless trial and error, it was coded as '*finding a general rule with Naïve Induction*' such as this answer 'Multiplying step number with 4 does not work. Multiplying it with 3 does not work, either. Yes, I can find the terms by multiplying the step number with 2 and subtracting 1. Then, this is the rule.'

If students verify their general rule on given terms, it was coded as *'Verification of the general rule'* such as this answer: 'Multiplying the step number with 2 and subtracting 1 gives the right answer at the first step. Multiplying the step number with 2 and subtracting 1 also gives the correct answer at the second step.'

If students calculate the terms of the pattern by applying the general rule, it was coded as '*Rote-calculation*' such as this answer: 'The general rule is 2n-1. To find the 50^{th} term, I will replace n with 50 and add 1. It is 99.'

3. Findings

The research question of the current study is about sixth-grade students' generalization process of patterns. The analysis of students' answers revealed three generalization approaches, as seen below.

- Approach 1 included (i) *generalizing near terms with AG*, (ii) *expressing the general rule with CG or SG*, (iii) *verification of the general rule*, and (iv) *rote calculation*;

- Approach 2 included (i) *generalizing near terms with AG*, (ii) *finding a general rule with naïve induction*, and (iii) *rote calculation;*
- Approach 3 included (i) *generalizing near terms with both AG and FG*, (ii) *expressing the general rule with CG and/or SG*, and (iii) *rote calculation (if needed).*

Table 1
The frequencies of each approach

	Question 1	*Question 2*	*Question 3*	*Frequency*
Approach 1	P1, P4	P3, P4	P1, P3, P4	7 times
Approach 2	-	P1	P2	2 times
Approach 3	P2, P3	P2	-	3 times

Among three of them, Approach 1 was the most observed model. It was seen seven times by one student (P1) in the first and third questions, by another student (P3) in the second and third questions, and by one student (P4) in all three questions. Approach 2 and Approach 3 were seen as much fewer than Approach 1. While Approach 2 was seen two times by one student (P1) in the second question and by another student (P2) in the third question, Approach 3 was seen three times by one student (P2) in the first and second questions and by another student (P3) in the first question (see Table 1).

3.1. Approach 1

The first approach, which was revealed as a result of the data analysis, included generalizing near terms with AG, expression of a general rule with CG or SG, verification of the general rule, and rote calculation. P1 and P4 in the first question, P3 and P4 in the second question, and P1, P3, and P4 in the third question followed this path. In this model, all students in all questions first translated figural patterns into numeric patterns. Then, they started generalizing the pattern to the near terms with arithmetical generalization. After the arithmetical generalization of near terms, they did not move to the factual generalization of near or far terms. Instead, they jumped to the 'find a general rule' part by skipping to generalize the pattern to far terms. With this intention, each student aimed to express a general rule in the form of d.n+b, where d represents the constant difference, n represents the step number, and a represents the constant.

When students' answers were analyzed, it was seen that these students were aware of the form of the general rule as 'the step number (n) times constant difference (d) plus/minus something (a)'. In other words, they firstly expressed the variable part by multiplying the step number with the constant difference. Then, they calculated the constant part by verifying their rule on the given steps. After completing this process, they continued to verify their general rule on the first few terms. When the verification is done, they rote-calculated the near and far terms, such as the 10^{th}, 50th, and 100th terms, by applying the general rule they formed, as exemplified below in Table 2.

As seen in Table 2, P3 noticed the constant difference between the terms of the pattern as 2 and used arithmetical generalization to reach the fourth and fifth terms. When asked about the tenth term, he needed a general rule and expressed the general rule as 2N+1. Then, he verified his rule on the first, second, and third terms. After verification, he rote-calculated the tenth and fiftieth steps. There is another example from the third question in Table 3.

In the example summarized in Table 3, P1 firstly noticed the constant difference as 3, then s/he expressed the general rule as "step number times 3 plus 2" and "n3+2". Then, s/he verified the rule on the first, second, third, fourth, and fifth steps. As the last step, s/he rote-calculated the number of circles in the tenth and fiftieth steps. There is another example from the first question presented in Table 4.

Table 2

An example from the second question (see Figure 3 for the original question)

Q2	
Arithmetical generalization	P3: (Student counts the number of circles in the first three steps and writes them upon the figures.) Here, the amounts [of circles] are increasing 2 by 2. In the fourth step, I should add 7 to 2; it is 9. To form the fifth step, I should add 2 again. 9 plus 2 equals 11. There are 11 circles in the fifth step.
Naïve Induction	-
Algebraic generalization	
Factual generalization	-
Contextual generalization	-
Symbolic generalization	P3: To form the tenth step, I should find the general rule. The general rule is increasing as 2N. But, it is becoming 2N+1.
Verification	P3: To find the first step, 2 times 1 is 2; 2 plus 1 is 3. 2 times 2 is 4; 4 plus 1 is 5. To find the third step, 2 times 3 is 6; 6 plus 1 is 7. They are all true.
Rote calculation	P3: 2 times 10 is 20; 20 plus 1 is 21. To find the fiftieth step, 50 times 2 is 100; 100 plus 1 is 101.

Table 3

An example from the third question (see Figure 4 for the original question)

Q3	
Arithmetical generalization	P1: There are 5 [matches] in the first step. There are 8 [matches in the second step] and 11 [matches in the third step]. Since it is increased by 3...
Naïve Induction	-
Algebraic generalization	
Factual generalization	-
Contextual generalization	P1: ...since it is increased by 3, it [the general rule] is the step number times 3 and plus 2.
Symbolic generalization	(P1 writes n3+2).
Verification	P1: When I multiplied 1 by 3 and added 2, it is 5 here [in the first step]. When I multiplied 2 by 3 and added 2, it is 8 [in the second step]. When I multiplied 3 by 3 and added 2, it is 11 [in the third step]. In the fourth step, 4 times 3 is 12 and 12 plus 2 is 14. In the fifth step, 5 times 3 is 15 and 15 plus 2 is 17…
Rote calculation	P1: So, [in order to calculate the tenth step] 10 times 3 is 30. 30 plus 2 is 32. In the 100^{th} step, 100 times 3 is 300, and 300 plus 2 is 302.

Table 4

An example from the first question (see Figure 2 for the original question)

Q1	
Arithmetical generalization	P4: This pattern increases by 3 here [points from the first term to the second term] and here [points from the second term to the third term]. Therefore, it should increase by 3 here [points to the fourth term]. So, this [the fourth term] is 9 plus 3.
Naïve Induction	-
Algebraic generalization	
Factual generalization	-
Contextual generalization	-
Symbolic generalization	P4: It is 3N…Yes, the pattern increases as 3N since they [terms] are increased by 3.
Verification	P4: For example, in the fourth step, when I replaced N with 4, it is 12. I had already found the 12.
Rote calculation	P4: [To find the tenth term] 3 times 10 is 30. [To find the fiftieth term] 3 times 50 is 150.

In this example (see Table 4), P4 firstly indicated the constant difference between the first three consecutive terms as 3, then s/he found the fourth term by adding the constant difference onto the third term. As the second step, s/he expressed the general rule as "3N" and verified the rule on the fourth term. After verification, s/he calculated the tenth and fiftieth terms.

3.2. Approach 2

The second approach, which was revealed as a result of data analysis, included generalizing near terms with AG, finding a general rule with NI, and rote calculation. P1 in the second question and P2 in the third question followed this path. In this model, all students first counted the figures and transformed them into numeric patterns. Then, they started generalizing the pattern to the near terms with arithmetical generalization. Similar to Approach 1, they directly intended to find a general rule as the second step. But, different from Approach 1, they used trial and error. When students' answers were analyzed, it was seen that these students were aware of the form of the general rule as 'the step number (n) times something plus/minus something'. Yet, as different from the students in the first model, these students were not aware of the necessity of multiplying the step number with the constant difference between consecutive terms. Therefore, they just made random guesses with the trial and error method, as seen in the answer of Participant 2 to the third question below in Table 5.

Table 5
An example from the third question (see Figure 4 for the original question)

Q3	
Arithmetical generalization	P2: There are 5 in the first step, 8 in the second step, and 11 in the third step. So, it increases by 3. 11 plus 3 is 14. 14 toothpicks are necessary for the fourth step. 14 plus 3 is 17. 17 toothpicks are required in the fifth step.
Naïve Induction	P2: It asks for the tenth step. Therefore, I need a rule. Let me try to multiply the step number by 2. [For the first step], 1 times 2 is 2; 2 plus 3 is 5. That is right for the first step. [For the second step], 2 times 2 is 4; 4 plus 3 is 7. That is not right for the second step. Let me try to multiply the step number with 3 and add 2. [For the first step], 1 times 3 is 3; 3 plus 2 is 5. It is right. [For the second step], 2 times 3 is 6; 6 plus 2 is 8. Yes, [the rule is] the step number times 3 plus 2. R: In the previous two questions, you said that you were multiplying the step number with the increase between the consecutive steps. Now, you first tried to multiply with 2. When it did not work, you tried to multiply with 3. Could you explain the reason? P2: It may not always be multiplied with the increment. I usually think of the trial and error method. Therefore, I firstly tried '2'. Since it did not work, I tried '3'. When I multiplied the fourth step by 3, it is 12; 12 plus 2 is 14. It still works.
Algebraic generalization	
Factual generalization	-
Contextual generalization	-
Symbolic generalization	-
Verification	-
Rote calculation	Now, it asks for the tenth step. 3 times 10 is 30; 30 plus 2 is 32. There are 32 toothpicks in the tenth step. [In the next item], it asks for the 100^{th} step. 100 times 3 is 300; 300 plus 2 is 302. There are 302 toothpicks in the 100^{th} step.

As seen in the example, P2 conducted near generalization with arithmetical generalization by generalizing the pattern to the fourth and fifth terms by adding the constant difference to the previous terms. After arithmetical generalization, P2 directly moved to the trial and error method and tried to multiply the step number with 2. When it did not work, P2 tried multiplying the step number with 3 and adding 2. When she tried the rule for the first and second terms, it was working. Then, she also tried the working rule on the fourth term and decided on it. As a last step, she rote-calculated the tenth and hundredth terms by applying the general rule. Below is another example from the second question in Table 6.

Table 6
An example from the second question (see Figure 3 for the original question)

Q2	
Arithmetical generalization	P1: In this question, it is increased by 2 in the second step and by 4 in the third step. R: Could you explain more? P1: I considered the first step at both (the second and third steps). I marked the first circle at both (steps). Here (in the first step), it is increased by 2. Here (in the second step), it is increased by 4. Here (in the third step), it is increased by 6. So, the fourth step will be 9. I mean, it will be increased by 8. There was 1 (showing the first circle); therefore, it will be 9. In the fifth step, I will add 10 to 1. It will be 11.
Naïve Induction	R: What about the tenth step? P1: With what should I multiply the step number? For example, step number times 3? No, it does not work here (the student shows the second step). Hmm… I guess I found it. Step number times 2 plus 1 works! For example, if I multiply 2 with 1 (student shows the first step) and add 1, I can find 3. If I multiply 2 with 2 (student shows the second step) and add 1, I can find 5. Here (student shows the third step), I can find 6 and add 1. It will be 7…. So, I multiplied by 2 and added 1. It worked at every step. Then, the tenth step will be 21, and the fiftieth step will be 101. The rule is the step number times 2 plus 1 (student writes n2+1).
Algebraic generalization	
Factual generalization	-
Contextual generalization	-
Symbolic generalization	-
Verification	-
Rote calculation	-

As understood from the example, P1 first reached arithmetical generalization and calculated the fifth step as 11. When s/he was asked about the tenth step, s/he looked for a general rule with naïve induction. First, s/he tried multiplying the step number with 3, but it was not a working rule. Then, s/he tried to multiply the step number with 2 and added 1. She tried the rule on the first few terms and became sure that it gave the right answers. Lastly, she calculated the tenth and fiftieth steps and expressed the general rule as "step number times 2 plus 1" and "n2+1".

3.3. Approach 3

The third approach, which was revealed as a result of the data analysis, included generalizing near terms with both AG and FG, expression of the general rule with CG and/or SG, and rote calculation (if needed). P2 and P3 in the first question and P2 in the second question followed this path. In this model, all three students first transformed figures into numbers and started generalizing the pattern to the near terms with arithmetical generalization. Then, they formed a factual generalization and reached some near terms with factual generalization. But, they also skipped the far term generalization. As a last step, they expressed the general rule, which they reached using factual generalization, with contextual or symbolic generalization. They did not need a verification step since they reached the general rule based on a factual generalization. Some of the students in Approach 3 used rote calculation.

Table 7
An example from the first question (see Figure 2 for the original question)

Q1	
Arithmetical generalization	P2: There are 3 circles in the first step, 6 in the second step, and 9 in the third step. The pattern increases by three [circles].
Naïve Induction	-
Algebraic generalization	
Factual generalization	P2: The first step equals 1 times 3. The second step times 3 equals 6. The third step times 3 equals 9. So, this is true.
Contextual generalization	P2: I can say that the general rule is the step number times 3.
Symbolic generalization	-
Verification	-
Rote calculation	P2: The first question asks the number of circles in the fourth step. I will multiply the fourth step with 4, it is 12. The second question asks the number of circles in the tenth step. I will multiply the tenth step with 3, it is 30. To calculate the number of circles in the fiftieth step, I will multiply the fiftieth step with 50, it is 150.

As exemplified above in Table 7 from the first question, P2 noticed the constant increase in the pattern with arithmetical generalization. Then, he noticed a different relationship in the first three terms and expressed the first term as 1 times 3 instead of just 3, the second term as 2 times 3 instead of just 6, and the third term as 3 times 3 instead of just 9. At this point, he reached a factual generality since he explored the mathematical structure of the first three terms. After factual generalization, he expressed the general rule with contextual generalization and rote-calculated the number of circles in the fourth, tenth, and fiftieth terms. In Table 8, another example is given from the second question.

Table 8
An example from the second question (see Figure 3 for the original question)

Q2	
Arithmetical generalization	P2: 3 circles in the first step and 5 in the second step. I expect 7 circles in the third step. Yes, right. Then, the fourth step becomes 9, and the fifth step becomes 11 since it increases by 2. It asks the number of circles in the tenth step. I can continue in the same way. 11 for the fifth step, 13 for the sixth step, 15 for the seventh step, 17 for the eighth step, 19 for the ninth step, and 21 for the tenth step…
Naïve Induction	-
Algebraic generalization	
Factual generalization	P2: Now, it asks for the number of circles in the fiftieth step. I need a rule. In the first step, 1 times 2 is 2, plus 1 equals 3. That is right. In the second step, 2 times 2 is 4, plus 1 equals 5.
Contextual generalization	P2: The rule is step number times 2 plus 1. R: How did you reach this rule? P2: There are 2 circles between the first and second steps. It increases by two [circles], so I multiplied by 2. Then, I had to add 1 to find the numbers.
Symbolic generalization	-
Verification	-
Rote calculation	P2: Now it asks for the fiftieth step. 50 times 2 is 100; plus 1 is 101.

As seen in this example, P2 first calculated the fourth, fifth, and tenth terms with arithmetical generalization. Then, s/he expressed the number of circles in the first and second steps with factual generalization. Lastly, she expressed the general rule with contextual generalization and calculated the number of circles in the fiftieth step.

4. Discussion

The present study aimed to explore sixth-grade students' pattern-generalization processes progressively. The results of the study revealed three generalization approaches. Through Approach 1, students first generalized near terms with arithmetical generalization; then, they did not generalize the pattern to far terms. More specifically, they expressed the general rule with contextual or symbolic generalization. After expressing the general rule, they verified it on given terms and rote-calculated far terms. Through Approach 2, students also began generalizing near terms with arithmetical generalization; then, they intended to find the general rule with trial and error. After seeing the general rule, they calculated far terms similar to Approach 1. Through Approach 3, the first step of students' generalization process was to reach near terms with arithmetic generalization, just like Approach 1 and Approach 2. However, as different from Approach 1 and Approach 2, these students did not jump to find/express the general formula as the second step. Instead, they formed a factual generalization and reached some near terms in this way. On the other hand, students with Approach 3 did not reach far term generalization, either. As the third step, they expressed the general rule either with contextual or symbolic generalization. Lastly, they calculated far terms just like others.

In light of these results, it was observed that all students followed a path from near term generalization to finding/expression of the general term and to rote-calculation of far terms. Actually, in the Pattern Questionnaire, each question had sub-questions, which were arranged from near generalization to far generalization and expression of the general terms sequentially based on Radford's (2000) generalization approach. Yet, it was revealed that students did not follow this path; after all, they were free to create their generalization process. While this result was consistent with some studies (Akkan & Çakıroğlu, 2012; Amit & Neria, 2008; Becker & Rivera, 2006), it was inconsistent with some of them (Varhol et al., 2021). The sequence of the generalization process is essential since it might give important clues about students' generalization skills. For example, when a student generally started with near generalization and continued with finding the n^{th} term by skipping far generalization, it can be interpreted that the student can conduct near generalizations, as s/he is not able to conduct far generalizations efficiently. This might indicate the possible gap that beginning algebra students experienced during the transition from arithmetic thinking to algebraic thinking. Therefore, in this study, students' general tendency to create a generalization process from near term to general term generalization by skipping far term generalization might show their possible gaps between arithmetic thinking and algebraic thinking.

It was also seen that all students generalized near terms with arithmetical generalization as a first step of their generalization process. This result was consistent with the related literature (Amit & Neria, 2008; Lannin, 2004; Lannin et al., 2006; Orton & Orton, 1999; Stacey & MacGregor, 2001). According to the literature, it is expected that students start generalizing patterns with additive reasoning through arithmetical generalization (Lannin, 2004) since it is easy to add the fixed difference to find the next term (Garcia-Cruz & Martinón, 1998). Additionally, it was shown that the nature of the patterning tasks leads students to additive thinking if the pattern is represented step by step (Barbosa & Vale, 2015; Lannin et al., 2006). Therefore, in the present study, students might find it useful and easy to apply arithmetical generalization as a first step. In addition, the step-by-step nature of patterning tasks might have led students to use additive relationship arithmetically.

Through students' generalization process, the absence of far term generalization was significant. As expressed in the findings section, all students, whatever their generalization approach is, firstly generalized the pattern to near terms and then looked for a general rule. When they found the general rule, they used it to calculate far terms by rote. The noticeable side of this result is the absence of generalizing the pattern to far terms with factual generalization. In other words, no student generalized the pattern to far terms algebraically; instead, they preferred calculating it by applying the general rule. While this result is consistent with some studies, which reported the absence of far term generalization (Amit & Neria, 2008; Ozdemir et al., 2015; Somasundram et al., 2019), it was inconsistent with some studies, which reported the existence of far term generalization through students' generalization processes (Cooper & Warren, 2011; Miller & Warren, 2012; Radford, 2003). Many studies showed the importance of far term generalization. Being able to generalize the pattern to far terms shows students' conceptual understanding of the nature of the generalization (Lannin et al., 2006). To be able to generalize a pattern to far terms, students need to reach factual generalization, since factual generalization necessitates exploring the mathematical structure of the pattern at the numerical level. To reach algebraic generalization, factual, contextual, and symbolic generalizations should follow each other (Radford, 2003). Thus, it is essential to enable students to engage with factual generalization to generalize the pattern to far terms. In this study, factual generalization was seen only three times, yet students used it for near term generalization, not far term generalization. This result might stem from two reasons. First of all, as expressed before, the Turkish middle school mathematics curriculum has two objectives related to pattern generalization; however, those objectives do not involve the process of generalizing the patterns to far terms. Instead, they focus on finding and expressing the general rule with letters (MoNE, 2018). As parallel to the curriculum, mathematics instruction dominantly focused on the procedures of constructing the general rule of the pattern (Lannin et al., 2006). Therefore, students cannot understand the algebraic structure of the patterns (Noss et al., 1997). Moreover, the pattern generalization questions in Turkish mathematics textbooks generally ask students to write the rule of the pattern before generalizing the pattern to far terms (Ayber, 2017). Therefore, the reason behind the absence of factual generalization of far terms might stem from not spending enough time for far term generalization during mathematics lessons by focusing on finding a general rule as advised in the curriculum. It might also stem from the textbook questions, which lead to finding a general rule right after finding the known terms.

It was also worth discussing that most students applied the standard procedure of finding a general rule of the pattern. In detail, all students translated all three figural pattern tasks into numerical patterns as a first step. Then, after the arithmetical generalization of near terms, students from Approaches 1 and 2 skipped the factual generalization of near and far terms and directly moved to find the general rule. Students from Approach 1 were aware of the standard procedure of multiplying the step number with a constant difference and adding or subtracting some number. As expressed in the results section, Approach 1 was the most observed approach. It was seen seven times. On the other hand, students from Approach 2 did not have the procedural knowledge of multiplying the step number with a constant difference; still, they were aware of the procedure of multiplying the step number with a number. This result showed consistency with past studies (Girit & Akyuz, 2016; Lithner, 2008; Maudy et al., 2018). Literature reported that beginning algebra students are mostly taught routine calculations of finding the general rule of the patterns in traditional classrooms (Lannin et al., 2006; Maudy et al., 2018). To conduct these routine calculations, traditional teachers mostly tend to place numerical patterns more than figural patterns into the class and to construct the general formula of the patterns by just counting (Vale & Cabrita, 2008). This is because traditional teachers did not have sufficient algebraical knowledge about pattern generalization (Demonty et al., 2018). Therefore, they generally prefer to translate a visual pattern into a numeric pattern (Vale & Cabrita, 2008). Yet, seeing a visual pattern is so important in constructing algebraic generalization. Lee and Wheeler (1987) stated that students experience difficulty in "seeing an algebraically useful pattern" by focusing on numerical aspect of patterns' structure (p.95). In a study (El Mouhayar & Jurdak, 2016), students used numerical approaches at near and far generalizations and figural approaches to generalizing the pattern to any term. Consistently, a study conducted by El Mouhayar (2018) reported a higher level of reasoning and generalization in the figural approach than the numerical approach in pattern generalization at each grade level. It was also reported that students with figural thinking are more tended to explore the algebraic relationship between terms and term numbers, while students with numeric thinking are more tended to trial and error with an inadequate understanding of the pattern's structure (Becker & Rivera, 2005). In the present study, students' translating figural patterns into numerical patterns and applying standard procedures of finding the general rule of the pattern might stem from students' habits of dealing with numerical patterns in procedure-focused classroom environments. Furthermore, students' tendency to translate figural patterns into numeric patterns might have prevented them from seeing the algebraic structure of the terms and led to the application of memorized standard procedures.

These findings are limited to the data collected from the beginning algebra students from one public school in Türkiye. Nevertheless, the generalization approaches of beginning algebra students might vary in different countries. Therefore, further research could be conducted to explore the generalization approaches of beginning algebra students at the international level. Furthermore, the results of the present study are limited to the sixth-grade students' trends of pattern-generalization approaches. Nonetheless, some of the past studies reported progressive development of students' algebraic reasoning across grade levels, while some studies found similar algebraic generalization structures regardless of grade levels at the middle school level. Thus, further study could be conducted with the 6th, 7th, and 8th-grade levels to see whether there is a progressive development across grade levels in Turkish scope.

In sum, it is believed that the results of the current study would expand the horizons of mathematics teachers and curriculum developers in some ways. Firstly, the middle school mathematics curriculum can allocate enough time for students to gain the objective of exploring the flow from the near term to the far term generalization as well as from the arithmetical generalization to algebraic generalization. Mathematics teachers can avoid creating a procedural instructional environment by focusing on the rote procedure of finding the general rule of a pattern. Also, patterns' figurative structures can be emphasized conceptually before the numerical structures during mathematics lessons. Teachers can enhance their knowledge of the transition from arithmetical to algebraic generalization through seminars or workshops.

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