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A Capacitated Lot Sizing Problem with Stochastic Setup Times

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Abstract. In this paper, we study a Capacitated Lot Sizing Problem with Stochastic Setup Times (CLSP-SST).We describe a mathematical model that considers both regular costs (including production, setup and inventory holding costs) and expected overtime costs (related to the excess usage of capacity). The CLSP-SST is formulated as a two-stage stochastic programming problem. A procedure is proposed to effectively compute the expected overtime for a given setup and production plan when the setup times follow a Gamma distribution. A sample average approximation scheme is used to obtain upper bounds and a statistical lower bound. This is then used to benchmark the performance of two additional heuristics. A first heuristic is based on changing the capacity in the deterministic counterpart, while the second heuristic artificially modifies the setup time. We conduct our computational experiments on well-known problem instances and provide comprehensive analyses to evaluate the performance of each heuristic.

Keywords: Lot sizing, stochastic setup times, overtime, sample average approximation, heuristics.

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1. Introduction

Lot sizing models aim to obtain the optimal production plan in which the timing and corresponding level of production are determined. In this paper, we focus on the Capacitated Lot Sizing Problem (CLSP) with overtime, which belongs to the class of *dynamic lot sizing problems* with a discrete time scale, finite time horizon and deterministic dynamic demand (see Pochet and Wolsey [32]). In the CLSP, several different items can be produced on a single machine in the same period. In its classical definition, this problem minimizes the total cost (including production, setup and inventory costs) incurred by a production plan that satisfies the demands and respects the time capacity of the machine. The production capacity inherently limits the resource consumption resulting from the setup (if considered) and production in each period. In most of the research on the CLSP, setup times are either ignored, or assumed to be deterministic. However, in practice it is possible that setup times are stochastic, and thus the quality of the solutions obtained by deterministic models may deteriorate once applied in real-life settings (leading to inefficient or even infeasible plans at the operational level).

Variability in setup times exists because of various reasons. It is possible that the setup process is not standardized, or that errors occur during this process, which increase the total setup time. In addition, there is always the inherent variability in the execution time of a specific activity. The literature on lean manufacturing emphasizes methods to reduce setup times and to provide work standardization (Forza [15] and McIntosh et al. [26]). These methods often include effective ways to reduce setup time variability as well. Doolen and Hacker [14] mention various practices such as the use of checklists during the setup of a machine, the use of documented standard operating procedures, the use of devices to reduce operator errors during the setup, and the use of training. In addition, McIntosh et al. [27] indicate that the lack of standard procedures may indeed lead to variable durations for the same type of setup. A detailed analysis of setup times given in a case study by Gilmore and Smith [16] also shows that waiting for key resources to perform the setup accounts for a large part of the total setup time. In some cases, setup times increase due to unplanned maintenance which needs to be performed in addition to the regular setup activities, such as the replacement of worn out tools (see McIntosh et al. [28]). In the latter paper, the authors provide an example of how poor equipment quality leads to delays in setup times. In conclusion, the above-mentioned literature indicates that variability in setup times is a realistic concern to companies.

In our problem setting, we consider a Capacitated Lot Sizing Problem with Stochastic Setup Times (CLSP-SST). In particular, we assume that setup times are stochastic following a given probability distribution. Consequently, we regard variability in setup times as a given and we do not seek to decrease it. Therefore, the aim of this paper is to obtain efficient production plans for the CLSP-SST. We adopt the static uncertainty strategy (see Bookbinder and Tan [6]), where setup and production decisions are fixed at the beginning of the planning horizon, and no dynamic adjustment of these decisions is executed. We assume that a company can use overtime at the end of any period if the given machine capacity is not sufficient due to the specific realizations of the setup times. The overtime values can be thought of as the recourse decisions, which are evaluated after observing the actual setup times. Stochastic setup times hence lead to extra costs in the form of overtime costs.

The contributions of this paper are fivefold:

- First, we introduce the lot sizing problem in the presence of stochastic setup times. To the best of our knowledge, no research has addressed this problem.
- Second, for a given setup and production plan, we develop an effective procedure to evaluate the expected overtime for the case where the setup times follow a Gamma distribution.
- Third, we propose a Sample Average Approximation (SAA) scheme to obtain upper bounds and a statistical lower bound.
- Fourth, we propose two effective heuristics that are easily applicable in practice. These heuristics are based on changing certain parameters in the deterministic counterpart of the problem.
- Fifth, comprehensive computational results using a standard data set allow us to compare the heuristic approaches and to provide managerial insights.

The remainder of the paper is organized as follows. In Section 2, we present a literature review which mostly focuses on the various stochastic versions of the lot sizing problems. In Section 3, we describe the model proposed for the CLSP-SST and discuss properties of the setup times and the expected overtime. In Section 4, we explain the SAA method and the two other heuristic methods proposed for the problem. In Section 5, we present the results obtained by solving the well-known problem instances of

Trigeiro et al. [39], which are adapted to include stochasticity in the setup times. Finally, in Section 6 we end the paper with our main findings and conclusions.

2. Literature Review

The CLSP (both the single-item version and the multi-item version) belongs to the class of NP-hard combinatorial optimization problems (see Bitran and Yanesse [5]), for which several effective optimal and heuristic solution procedures have been proposed in the literature. Karimi et al. [21] review single-level lot sizing problems, their extensions and available (exact and heuristic) solution approaches. The interested reader is referred to Brahimi et al. [7] for a review on single-item lot sizing problems, where both uncapacitated and capacitated variants are considered. Jans and Degraeve [18, 19] present comprehensive reviews of the available mathematical formulations, and solution approaches for several dynamic lot sizing problems. Buschkühl et al. [9] focus on models and algorithms proposed for multi-item dynamic lot sizing problems, where various exact and approximate algorithms are reviewed.

Stochastic versions of lot sizing problems mostly focus on demand uncertainty (see Tempelmeier [37] and Aloulou et al. [1] for recent reviews on stochastic lot sizing problems). Bookbinder and Tan [6] employ α-service level constraints (where α denotes the probability that inventory is non-negative), and develop three strategies to handle the resulting setting: static uncertainty, dynamic uncertainty and static-dynamic uncertainty. The static uncertainty strategy applies the idea of having frozen schedules, in which all decisions related to setups and production levels are determined at the beginning of the planning horizon. This decision rule, which has been widely used in the literature for stochastic lot sizing problems (see, e.g., Tempelmeier [36], and Tempelmeier and Hilger [38]), is also the main strategy that we consider in our paper. Jeunet and Jonard [20] develop a number of lot sizing techniques to assess the effects of demand variability on production plans with respect to two criteria: the regular cost-effectiveness and robustness. Results, which are obtained by simulation procedures, indicate that these two criteria are negatively correlated. Brandimarte [8] represents the demand uncertainty by a directed scenario tree, which corresponds to a multi-stage mixed-integer stochastic programming model with recourse. The proposed formulation is based on a plant-location model, and a heuristic method employing the fix-and-relax strategy introduced by Dillenberger et al. [13] is developed. Results show that considering demand uncertainty in the planning phase provides significant improvements especially for the instances with tight capacity. Recently, Koca et al. [23] focus on a capacitated lot sizing problem with stochastic demands and controllable processing times. Specifically, the demand of each period follows a Normal distribution and processing times can be reduced by outsourcing, adjusting the machine speed and so on. The reduction in processing time entails the so-called compression costs. It is further assumed that the compression cost function is convex. The proposed problem is solved by considering the static uncertainty strategy and α-service level constraints introduced by Bookbinder and Tan [6]. Results indicate that controllable processing times are more effective for the instances with medium capacity and high setup costs. Rossi et al. [33] consider a stochastic version of the lot sizing problem in which the demand is assumed to be uncertain and non-stationary. The authors develop a unified mixed-integer linear programming model based on a piecewise linear approximation of the loss function. The proposed models are based on the static-dynamic uncertainty strategy. Results show that the solution approach is flexible and effective, and it computes an accurate estimation of the total expected cost.

The survey presented in Aloulou et al. [1] indicates that stochastic variants of lot sizing problems are mostly based on single-item single-period single-machine problems. Beraldi et al. [3] consider a version which is the lot sizing and scheduling problem with identical parallel machines and stochastic processing times. In this problem, setup costs are assumed to be sequence-dependent and stochastic parameters are modelled by using a scenario tree. A multi-stage mixed-integer stochastic programming formulation is proposed and efficient heuristic procedures based on the fix-and-relax strategy are developed. Dellaert and Melo [11] focus on a variant which is a stochastic single-item production system in a make-to-order environment. They extend existing strategies, which have been proposed for similar stochastic versions, by employing overtime costs whenever the total production exceeds the given capacity. The latter is consumed only by production, i.e., setup times are not considered. In the proposed setting, stochasticity is incurred due to the partial customer-order information. A number of procedures are developed to obtain near-optimal production lot sizes. These procedures are mainly based on the standard (s, S) and (R, S) policies used in make-to-stock problems.

Dynamic lot sizing models mostly assume that items are produced on reliable machines. Kuhn [25] focuses on a single-item uncapacitated lot sizing problem with stochastic machine breakdowns. Two cases are considered: (i) the setup is totally lost after a breakdown and a new setup is required, and

(ii) the cost for the setup needed to resume the production of the same item after a breakdown is much lower than the original setup cost. A stochastic dynamic programming model is developed to obtain an optimal production plan for the proposed policies. Nourelfath [29] studies a multi-period multi-item capacitated lot sizing problem where machine breakdowns are assumed to be stochastic and the capacity is consumed only by production, i.e., setup times are not considered. The proposed model includes a set of constraints to ensure some minimum probability of meeting the customer service level within a pre-determined value, and it is solved by a two-phase solution approach.

In this paper, we study the CLSP-SST where setup times are random variables with a known probability distribution. The aim in our problem setting is to minimize the total cost including regular production costs and expected overtime costs incurred due to the excess usage of the capacity. This problem has not been discussed before and the literature review indicates that uncertainty in lot sizing problems is mainly focused on other problem characteristics such as demands and machine breakdowns.

3. Problem Statement and Model Formulation

We first provide the formal definition of the classical CLSP with overtime, and next provide the formulation of the CLSP-SST in Section 3.1. Let $P = \{1, ..., n\}$ be the set of items and $T = \{1, ..., m\}$ be the set of time periods. Several items can possibly be produced in the same time period. This is the basic assumption in a big bucket lot sizing model. A single machine is used for all items and this machine has a limited capacity C_t in each period t. Each time that the production begins for item i in period t, a setup takes place with a cost sc_i by using st_i units of capacity (setup time). Producing one unit of item i entails a cost vc_i and consumes vt_i units of capacity (unit production time). A penalty cost oc_t is associated with the use of overtime resulting from a capacity violation at time period t (see Özdamar and Birbil [31], Özdamar and Barbarosoğlu [30], and Barbarosoğlu and Özdamar [2]). A cost hc_i is incurred for each end-of-period inventory of item i. Moreover, the demand of each item i in each period t (d_{it}) is known. The standard formulation of the CLSP with overtime (Model-SF) is given as follows:

$$\text{(Model-SF)} \quad \min \sum_{i \in P} \sum_{t \in T} (vc_i x_{it} + sc_i y_{it} + hc_i s_{it}) + \sum_{t \in T} oc_t o_t \tag{1}$$

$$\begin{aligned}
\text{subject to} \quad & s_{i,t-1} + x_{it} = d_{it} + s_{it}, && i \in P, t \in T, && (2)\\
& x_{it} \leq M y_{it}, && i \in P, t \in T, && (3)\\
& \sum_{i \in P} (vt_i x_{it} + st_i y_{it}) \leq C_t + o_t, && t \in T, && (4)\\
& s_{it} \geq 0, && i \in P, t \in T, && (5)\\
& x_{it} \geq 0, && i \in P, t \in T, && (6)\\
& o_t \geq 0, && t \in T, && (7)\\
& y_{it} \in \{0, 1\}, && i \in P, t \in T. && (8)
\end{aligned}$$

In the above model, x_{it} represents the number of units of item i produced in period t, y_{it} is the binary setup variable associated with item i in period t, s_{it} is the number of units of item i in stock at the end of period t, and o_t is the amount of overtime used in period t. The objective (1) is to minimize the total cost of production, setup, inventory and overtime. Constraints (2) correspond to the material balance equations: for each item i, the demand in period t is satisfied by the production in the current period t together with the inventory from the previous period, and the excess amount builds up the inventory for the next period. Constraints (3) indicate that production of an item in a period requires a setup in that period. In these setup constraints, big-M is a sufficiently large constant which can be set to $\sum_{j=t}^{m} d_{ij}$. In other words, production cannot exceed the remaining demand (the total demand that should be satisfied in periods t to m). Constraints (4) ensure that the total time spent for production and setup in each period does not exceed the capacity limit plus the used overtime. Constraints (5) are needed since we do not allow backorders, and constraints (6)–(8) specify the domain of production, overtime and setup variables, respectively.

3.1. The Capacitated Lot Sizing Problem with Stochastic Setup Times

In the CLSP-SST, setup times are random variables with a known probability distribution. The production quantities and setups are decided before the actual setup times are known, and these decisions remain fixed. As discussed, this setting with a frozen schedule corresponds to the static uncertainty strategy given by Bookbinder and Tan [6]. The actual setup times are only revealed when doing the setup. The company does not change its production plan after the setup times are known, but resorts to overtime usage if the current capacity in a period is not sufficient. Therefore, we account for the expected overtime cost in the objective function. The model proposed for this problem (Model-SST) is as follows:

$$\text{(Model-SST)} \quad \min \sum_{i \in P} \sum_{t \in T} (vc_i x_{it} + sc_i y_{it} + hc_i s_{it}) + \sum_{t \in T} oc_t O_t(\mathbf{x}_t, \mathbf{y}_t) \quad (9)$$

$$\begin{aligned} \text{subject to} \quad & s_{i,t-1} + x_{it} = d_{it} + s_{it}, && i \in P, t \in T, && (10) \\ & x_{it} \leq M y_{it}, && i \in P, t \in T, && (11) \\ & s_{it} \geq 0, && i \in P, t \in T, && (12) \\ & x_{it} \geq 0, && i \in P, t \in T, && (13) \\ & y_{it} \in \{0, 1\}, && i \in P, t \in T. && (14) \end{aligned}$$

In this model, the vectors $\mathbf{x}_t$ and $\mathbf{y}_t$, where $\mathbf{x}_t = \{x_{it} \mid i \in P\}$ and $\mathbf{y}_t = \{y_{it} \mid i \in P\}$, are used to denote the production quantities and setup variables of each item in period t, respectively. Let $O_t(\mathbf{x}_t, \mathbf{y}_t)$ denote the expected overtime resulting from the production plan $(\mathbf{x}_t, \mathbf{y}_t)$. The objective is to minimize the total cost which consists of two main components: (i) the total cost of production, setup and inventory, and (ii) the total cost of expected overtime of the capacity. The latter component is imposed in the model by considering the capacity as a soft constraint, which means that the capacity constraint (4) is omitted in the model. However, the capacity constraint is implicitly taken into account through the calculation of the expected overtime $O_t(\mathbf{x}_t, \mathbf{y}_t)$. In other words, $O_t(\mathbf{x}_t, \mathbf{y}_t)$ depends on the production and setup plan, and the capacity parameters (i.e., unit production times, stochastic setup times and capacity). The related calculations are described in Section 3.2 by considering the Gamma distribution. In this model, we keep the demand (10), setup (11), and domain constraints (12)–(14). This formulation is a two-stage stochastic programming with recourse (see Birge and Louveaux [4]). The production and setup decisions are the first-stage decisions, and the overtime values correspond to the complete recourse decisions in the second stage.

3.2. Properties of the Setup Times and the Expected Overtime

In this paper, we model setup times by employing the Gamma distribution. This distribution is suitable for our problem since it is limited to non-negative values. In the Gamma distribution, large values are possible but with small probabilities. This is a reasonable assumption for stochastic setup times. The Gamma distribution also comprises several distributions as a special case (such as Exponential, Erlang and Chi-Square), and its additive property plays an important role in the computations of the expected values. Given the Gamma distribution for the setup times, we develop an analytical expression to exactly calculate the expected overtime for a given production and setup plan. The heuristics developed in this paper (see Section 4) can still be applied if another distribution function is chosen for the setup times, but in that case the calculation of the expected overtime for a given production and setup plan might need to be done via simulation.

Let S_{it} be the random setup time for item i in period t. We assume that S_{it} is Gamma distributed with shape parameter αst_i and scale parameter λ. The mean and variance of S_{it} are computed as follows:

$$E[S_{it}] = \alpha\lambda st_i, \tag{15}$$

$$\text{Var}(S_{it}) = \alpha\lambda^2 st_i. \tag{16}$$

In this paper, we impose that $\alpha\lambda = 1$, so that the expected setup time for each item i in period t ($E[S_{it}]$) is equal to its deterministic setup time (st_i). Note that this is needed to fairly compare the solutions obtained by the proposed model to those obtained by the deterministic models.

The total time spent in period t to setup the machine for all products produced in that period (S_t) is then defined as follows:

$$S_t = \sum_{i \in P_t} S_{it}, \tag{17}$$

where P_t denotes the set of items produced in period t, $P_t = \{i \mid y_{it} = 1, i \in P\}$. It is easy to observe from Equation (17) that S_t is the sum of random setup times of each item produced in period t. These variables have the same scale parameter, and thus S_t is also Gamma distributed. The shape parameter α_t and the scale parameter λ_t of S_t are then represented as follows:

$$\alpha_t = \alpha \sum_{i \in P_t} st_i, \tag{18}$$

$$\lambda_t = \lambda. \tag{19}$$

If $C_t > \sum_{i \in P_t} vt_i x_{it}$ (the time capacity is greater than the total time spent for production in period t), the expected overtime is calculated as follows (see Dellaert et al. [10] and Taş et al. [34, 35] for similar procedures):

$$\begin{aligned} O_t(\mathbf{x}_t,\mathbf{y}_t) &= \int_{C'_t}^{\infty} (u - C'_t)\frac{(e^{-u/\lambda_t})u^{\alpha_t-1}}{\Gamma(\alpha_t)(\lambda_t)^{\alpha_t}}du, \\ &= \int_{C'_t}^{\infty} \frac{(e^{-u/\lambda_t})u^{\alpha_t}}{\Gamma(\alpha_t)(\lambda_t)^{\alpha_t}}du - C'_t\int_{C'_t}^{\infty} \frac{(e^{-u/\lambda_t})u^{\alpha_t-1}}{\Gamma(\alpha_t)(\lambda_t)^{\alpha_t}}du, \\ &= \alpha_t\lambda_t\Big(1-\Gamma_{\alpha_t+1,\lambda_t}(C'_t)\Big) - C'_t\Big(1-\Gamma_{\alpha_t,\lambda_t}(C'_t)\Big), \end{aligned} \tag{20}$$

where $C'_t = C_t - \sum_{i\in P_t} vt_i x_{it}$ (i.e., the remaining capacity). Note that we use the remaining capacity in the computation of the expected overtime since the total production time for each product, i.e., $vt_i x_{it}$, is deterministic for a given production plan.

If $C_t \leq \sum_{i\in P_t} vt_i x_{it}$, $O_t(\mathbf{x}_t,\mathbf{y}_t)$ is calculated by:

$$O_t(\mathbf{x}_t,\mathbf{y}_t) = E[S_t] + \Big(\sum_{i\in P_t} vt_i x_{it}\Big) - C_t. \tag{21}$$

In the following proposition, we present the convexity property of the expected overtime.

Proposition 3.1. *The expected overtime in period t $O_t(\mathbf{x}_t,\mathbf{y}_t)$ is a convex function of the capacity consumption for that period in the deterministic counterpart, i.e., $\sum_{i\in P}(vt_i x_{it} + st_i y_{it})$.*

Proof. Suppose that a production plan is generated for a period t where the production quantities and the corresponding setup decisions are known. For the sake of simplicity, we refer to O_t as the expected overtime in that period computed with respect to the given production plan. Let u denote the capacity consumption for additional production that takes place in period t while keeping the values of the setup variables unchanged. In other words, the capacity consumption is increased by u. To prove that the expected overtime is a convex function of the capacity consumption in period t, we need to show that O_t is a convex function of u. More specifically, we must prove that $\frac{\partial^2 O_t}{\partial u^2} \geq 0$ by using the fact that O_t is continuously differentiable. We distinguish between two cases:

Case 1. $C'_t - u > 0$.

O_t is computed based on Equation (20) as follows:

$$O_t = \alpha_t\lambda_t\Big(1-\Gamma_{\alpha_t+1,\lambda_t}(C'_t-u)\Big) - (C'_t-u)\Big(1-\Gamma_{\alpha_t,\lambda_t}(C'_t-u)\Big).$$

It is known from the properties of the Gamma function that:

$$\begin{aligned}\Gamma_{\alpha,\lambda}(q) =& \frac{1}{\Gamma(\alpha)}\int_0^q \frac{e^{-z/\lambda}z^{\alpha-1}}{\lambda^\alpha}dz\\ =& \frac{1}{\Gamma(\alpha)}\int_0^{\frac{q}{\lambda}} e^{-y}y^{\alpha-1}dy\\ =& \frac{1}{\Gamma(\alpha)}\gamma(\alpha,\frac{q}{\lambda}).\end{aligned} \tag{22}$$

In Equation (22), $\gamma(\alpha, \frac{q}{\lambda})$ corresponds to the lower incomplete Gamma function with parameters α and $\dfrac{q}{\lambda}$, where $\alpha \geq 0$, $q \geq 0$ and $\lambda > 0$. The first and second derivative of $\gamma(\alpha, \frac{q}{\lambda})$ with respect to q are then computed as follows:

$$\frac{\partial\gamma(\alpha,\dfrac{q}{\lambda})}{\partial q} = \frac{1}{\lambda}\left(\frac{q}{\lambda}\right)^{\alpha-1} e^{-\frac{q}{\lambda}} \quad \text{and} \tag{23}$$

$$\frac{\partial^2\gamma(\alpha,\dfrac{q}{\lambda})}{\partial q^2} = \frac{1}{\lambda^2}\left(\frac{q}{\lambda}\right)^{\alpha-2} e^{-\frac{q}{\lambda}}\left(\alpha-1-\frac{q}{\lambda}\right). \tag{24}$$

Then, $\dfrac{\partial^2 O_t}{\partial u^2}$ is computed as follows:

$$\begin{aligned}\frac{\partial^2 O_t}{\partial u^2} =& -\alpha_t\lambda_t\frac{\partial^2\gamma\left(\alpha_t+1,\frac{C'_t-u}{\lambda_t}\right)}{\partial u^2}\frac{1}{\Gamma(\alpha_t+1)} \quad + \quad (C'_t-u)\frac{\partial^2\gamma\left(\alpha_t,\frac{C'_t-u}{\lambda_t}\right)}{\partial u^2}\frac{1}{\Gamma(\alpha_t)}\\ &-2\frac{\partial\gamma\left(\alpha_t,\frac{C'_t-u}{\lambda_t}\right)}{\partial u}\frac{1}{\Gamma(\alpha_t)}.\end{aligned} \tag{25}$$

By incorporating Equations (23) and (24) into Equation (25), we obtain:

$$\begin{aligned}\frac{\partial^2 O_t}{\partial u^2} =& -\alpha_t\lambda_t\frac{1}{\lambda_t^2}\left(\frac{C'_t-u}{\lambda_t}\right)^{\alpha_t-1} e^{-\left(\frac{C'_t-u}{\lambda_t}\right)}\left(\alpha_t-\left(\frac{C'_t-u}{\lambda_t}\right)\right)\frac{1}{\Gamma(\alpha_t+1)}\\ &+(C'_t-u)\frac{1}{\lambda_t^2}\left(\frac{C'_t-u}{\lambda_t}\right)^{\alpha_t-2} e^{-\left(\frac{C'_t-u}{\lambda_t}\right)}\left(\alpha_t-1-\left(\frac{C'_t-u}{\lambda_t}\right)\right)\frac{1}{\Gamma(\alpha_t)}\\ &+2\frac{1}{\lambda_t}\left(\frac{C'_t-u}{\lambda_t}\right)^{\alpha_t-1} e^{-\left(\frac{C'_t-u}{\lambda_t}\right)}\frac{1}{\Gamma(\alpha_t)}.\end{aligned} \tag{26}$$

Since $\Gamma(\alpha_t+1)=\alpha_t\Gamma(\alpha_t)$, Equation (26) leads to:

$$\frac{\partial^2 O_t}{\partial u^2} = \frac{1}{\lambda_t^{\alpha_t}}\left(C_t' - u\right)^{\alpha_t - 1} e^{-\left(\frac{C_t'-u}{\lambda_t}\right)} \frac{1}{\Gamma(\alpha_t)}. \tag{27}$$

We observe from Equation (27) that $\frac{\partial^2 O_t}{\partial u^2} > 0$ for any u, where $u \geq 0$ and $u < C_t'$.

Case 2. $C_t' - u \leq 0$.

O_t is computed by employing Equation (21) as follows:

$$O_t = E[S_t] - (C_t' - u). \tag{28}$$

It is easy to observe from Equation (28) that $\frac{\partial O_t}{\partial u} > 0$ and $\frac{\partial^2 O_t}{\partial u^2} = 0$.

These two cases show that for any period t, O_t is a convex function of u. Therefore, we can conclude that for any period, the expected overtime is a convex function of the capacity consumption in that period.

□

To illustrate the convexity property just proven, we solve one instance (X11227B) from Trigeiro et al. [39] by the SAA method (the details of this method will be presented in Section 4.1). We focus on one time period, and we gradually increase the consumption of the capacity by producing more units of some or all items that are already produced in that period. Figure 1 presents the detailed results. From this figure, it is clear that the expected overtime increases exponentially when we get closer to 100% capacity consumption. This insight will be used in the construction of our two heuristics.

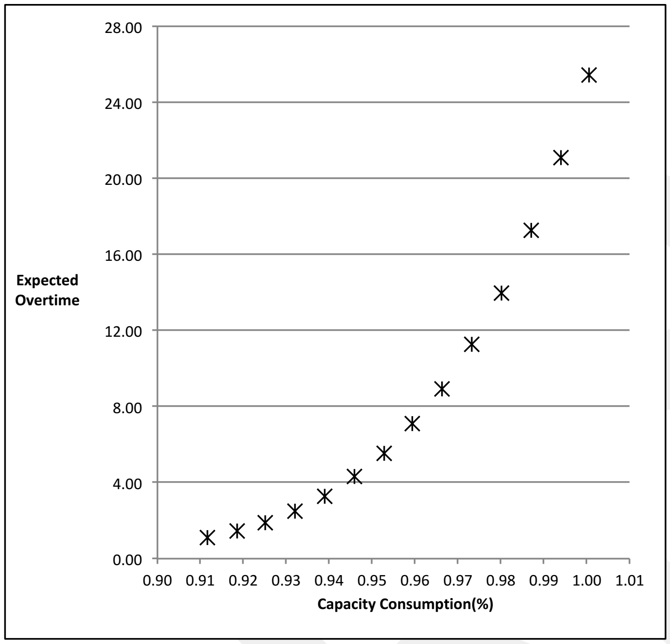

Figure 1: The capacity consumption (in percentage) and the corresponding expected overtime values in one time period ($t = 17$) for instance X11227B

4. Solution Methods

We first present the SAA method, which yields upper bounds and a statistical lower bound, in Section 4.1. This lower bound is used to evaluate the quality of the solutions generated by the two other heuristics presented in Section 4.2.

4.1. Sample Average Approximation

The SAA method (Verweij et al. [40]) is based on solving R replications of the stochastic problem, each of which takes into account only a limited set K of scenarios, sampled from the original distributions. Assuming each replication is solved to optimality, the average objective function value of the R problems provides a statistical lower bound to the original problem. The optimal solutions to the R problems are then reevaluated under a larger set of scenarios to obtain an estimation of their true objective function values. The solution achieving the lowest estimated cost is assumed to be the best upper bound found to the original problem. Furthermore, the SAA method specifies the computation of the estimated variance of the gap estimator. Further details can be found in Verweij et al. [40].

In what follows, we provide details on our implementation of the SAA method for the CLSP-SST. The stochastic formulation considering a limited set K of sample scenarios (Model-SAA) is defined as follows:

$$\text{(Model-SAA)} \quad \min \sum_{i \in P} \sum_{t \in T} (vc_i x_{it} + sc_i y_{it} + hc_i s_{it}) + \sum_{t \in T} oc_t \sum_{k \in K} \Big(\frac{1}{|K|} o_t^k \Big) \tag{29}$$

$$\begin{aligned}
\text{subject to} \quad & s_{i,t-1} + x_{it} = d_{it} + s_{it}, && i \in P, t \in T, && (30)\\
& x_{it} \leq M y_{it}, && i \in P, t \in T, && (31)\\
& \sum_{i \in P} (vt_i x_{it} + st_i^{k,t} y_{it}) \leq C_t + o_t^k, && k \in K, t \in T, && (32)\\
& s_{it} \geq 0, && i \in P, t \in T, && (33)\\
& x_{it} \geq 0, && i \in P, t \in T, && (34)\\
& y_{it} \in \{0,1\}, && i \in P, t \in T, && (35)\\
& o_t^k \geq 0, && k \in K, t \in T. && (36)
\end{aligned}$$

In this model, $st_i^{k,t}$ denotes the realized setup time of item i in period t according to scenario k, and o_t^k is the decision variable expressing the overtime used in period t upon the realization of scenario k. The objective is to minimize the total cost of production, setup and inventory plus the average cost incurred for overtime over all sample scenarios in K.

The SAA model is solved R times with different sets K of sample scenarios. Let $z_1, \ldots, z_R$ denote the optimal objective function values of the R solutions obtained by the SAA model. It has been shown (see Kleywegt et al. [22]) that

$$\underline{z} = \frac{\sum_{r=1}^{R} z_r}{R} \tag{37}$$

provides a statistical estimate for a lower bound on the optimal value of the original problem, i.e., as expressed in the formulation (9)–(14). Given that the SAA model in our case could be challenging to solve, it is possible that an optimal solution for the SAA model is not found within a preset time limit. In such cases, we use the lower bound as a substitute for the optimal objective function values of the SAA model. Therefore, $\underline{z}$ remains a valid statistical estimate for a lower bound on the optimal value of the original problem.

In order to provide a statistical estimate for the upper bound on the optimal value of the original problem, the SAA method approximates the true cost of each of the R solutions. The approximation is obtained by using a new sample set, which contains a much larger number of sample scenarios compared to K. Assuming Gamma distributed random setup times, we are able to calculate the true expected overtime for a given solution (as described in 3.2). Therefore, for a given solution, we are able to exactly compute its objective function value according to the original problem (i.e., Equation 9). We denote this value by z_r^*.

Thus, our upper bound is

$$\bar{z} = \min\{z_1^*, \ldots, z_R^*\}. \tag{38}$$

The previous calculations allow us to compute the following estimate for the absolute gap, $\bar{z} - \underline{z}$. Furthermore, the SAA method allows the calculation of the variances of $\underline{z}$ and $\bar{z}$. Since we are able to exactly evaluate the upper bound, its variance is zero. The variance of the $\underline{z}$, is defined as follows (see Verweij et al. [40]):

$$\hat{\sigma}_{\underline{z}}^2 = \frac{1}{(R-1)R} \sum_{r=1}^{R} (z_r - \underline{z})^2. \tag{39}$$

The variance of the absolute gap estimator is the sum of the variance of the lower bound estimate and the variance of the upper bound estimate. Since the latter is zero, the variance of the absolute gap estimator in our case is $\hat{\sigma}_{\underline{z}}^2$.

4.2. Heuristics H1 and H2

The two heuristics that we propose in this section are based on Proposition 3.1. This provides the insight that when we are close to 100% capacity consumption, even a small decrease in the capacity consumption provides a substantial decrease in the expected overtime. Both heuristics are based on the idea of solving the deterministic CLSP with overtime considering modified problem parameters and then evaluating the obtained solution with respect to stochastic setup times to obtain the true expected cost. In the first heuristic H1, we decrease the available capacity in the deterministic model, and as such we introduce some buffer capacity, and hence obtain lower expected overtime when the resulting solution is evaluated in the stochastic setting with the true capacity level. In the second heuristic H2, we increase the setup times used in the deterministic model with similar effects.

5. Computational Experiments

We conduct our computational experiments on the well-known data sets developed by Trigeiro et al. [39] for the CLSP with deterministic setup times. In these sets, the values of five problem parameters vary as follows: number of items (10, 20 or 30), setup time (low or high), setup cost (low, medium or high), demand variability (medium or high) and capacity utilization (low, medium or high). Trigeiro et al. [39] randomly generate five instances for each class, leading to 540 problem instances in total. Moreover, within each instance each period is assumed to have the same capacity. Note that in the original Trigeiro et al. [39] instances the unit production costs are set to zero. This stems from the fact that all demand must be satisfied and the production costs are time invariant, so that the total production cost is fixed.We set the parameters of the Gamma distribution for the setup times as follows: $\alpha = 1$ and $\lambda = 1$. To generate an overtime cost which is in line with other costs in the data set, we calculate the total holding and setup cost in each period using an EOQ cost approximation. The obtained total cost is then divided by the total capacity. More specifically, for each instance, oc_t is computed as follows:

$$oc_t = \rho\left(\frac{\sum_{i \in P} \sqrt{2\overline{d_i} sc_i hc_i}}{C_t}\right). \tag{40}$$

We conducted a number of preliminary experiments to determine an appropriate value of ρ. In the above expression, ρ is set to 50 according to the results of these experiments. Parameter $\overline{d_i}$ represents the average demand of item i and is calculated by:

$$\overline{d_i} = \frac{\sum_{t \in T} d_{it}}{m}. \tag{41}$$

Each model described in this paper is coded in C++ and solved using IBM ILOG CPLEX 12.5 [17]. All experiments are performed on an Intel(R) Xeon(R) CPU X5675 with 12-Core 3.07 GHz and 96 GB of RAM (by using a single thread). We set the computational time limit to 30 minutes.

5.1. Reformulation of the CLSP with Overtime

In the literature, several reformulations of the classical lot sizing models are provided (see Pochet and Wolsey [32] for an overview). Denizel and Süral [12] reported that the transportation problem reformulation, which was originally proposed by Krarup and Bilde [24], performs the best among the presented models. It is well-known that this reformulation provides a better LP relaxation gap compared to the formulation in the original variables (see, e.g., Pochet and Wolsey [32]). The transportation reformulation adapted to include overtime (Model-TR) is presented as follows:

$$\text{(Model-TR)} \quad \min \sum_{i\in P}\Big(vc_i \sum_{t\in T}\sum_{l=t}^{m} z_{itl} + sc_i \sum_{t\in T} y_{it} + hc_i \sum_{l\in T}\sum_{t=1}^{l-1}(l-t)z_{itl}\Big) + \sum_{t\in T} oc_t o_t \tag{42}$$

subject to

$$\sum_{t=1}^{l} z_{itl} = d_{il}, \qquad i\in P, l\in T, \tag{43}$$

$$z_{itl} \leq d_{il} y_{it}, \qquad i\in P, t\in T, l=t,...,m, \tag{44}$$

$$\sum_{i\in P}\Big(\sum_{l=t}^{m} vt_i z_{itl}\Big) + st_i y_{it} \leq C_t + o_t, \qquad t\in T, \tag{45}$$

$$z_{itl} \geq 0, \qquad i\in P, t\in T, l=t,...,m, \tag{46}$$

$$o_t \geq 0 \qquad t\in T \tag{47}$$

$$y_{it} \in \{0,1\}, \qquad i\in P, t\in T. \tag{48}$$

In this model, z_{itl} denotes the number of units of item i produced in period t to satisfy the demand in period l, where $l \geq t$. The objective (42) is to minimize the total cost of production, setup, inventory and overtime. Constraints (43) ensure that the demand of item i in period l is satisfied by the total amount produced in periods 1 through l. Constraints (44) indicate that a setup is required for each production run. Constraints (45) state that the total time spent for production and setup cannot exceed the capacity and the overtime. Constraints (46)–(48) impose non-negativity for production and overtime variables and integrality for setup variables, respectively.

We first compare the performance of the standard formulation (Model-SF) to that of the transportation problem reformulation (Model-TR). Specifically, we obtain solutions for all 540 problem instances by solving these two models setting all setup times at their average values, and next evaluating this solution in the stochastic setting to obtain the true expected total costs. In Table 1, we first present performance indicators pertaining to the deterministic models in columns 2–5: the number of instances solved to optimality (Opt), the average total cost (TC) of setup, inventory and overtime, the average final optimality gap in percentage (Gap(%)) and the average computation time in seconds. In columns 6–8, we summarize the results of the evaluation with respect to stochastic setup times: the average total expected overtime (EO), the average total expected overtime cost (EOC), and the average total expected cost (TEC) which includes setup, inventory and expected overtime cost. Results given in Table 1 indicate that Model-TR performs better than Model-SF both in terms of the computation time and of the solution quality. These results are in line with those of Denizel and Süral [12] for the deterministic problem. As a consequence, we use the transportation problem reformulation in the remainder of our computational experiments and the Model-SAA presented in Section 4.1 is also reformulated according to this transportation model.

Table 1: Results of Model-SF and Model-TR with $\alpha = 1$, $\lambda = 1$ and $\rho = 50$

	Deterministic Solution				Evaluation		
Model	Opt	TC	Gap(%)	Seconds	EO	EOC	TEC
SF	366	64180.40	0.31	639.94	29.31	2662.33	66842.00
TR	370	64154.80	0.25	613.99	29.10	2656.40	**66811.20**

The solution obtained by Model-TR using the average values for the setup times and its stochastic evaluation, constitute a naive baseline heuristic solution. In what follows, we benchmark the performance of our proposed heuristics to this baseline heuristic.

5.2. The Sample Average Approximation Heuristic

The quality of the SAA heuristic depends on the number of sample scenarios, i.e., $|K|$, as well as the number of replications R. We first test different values of $|K|$, and then we choose the value yielding the lowest total expected cost. For that value, we then examine the effects of using various values of R.

Table 2 presents the results of experiments where the number of scenarios $|K|$ ranges from 10 to 100. Note that for each $|K|$, only one run is carried out and an experiment with several replications will be discussed next. The columns in Table 2 give the same information as in Table 1, except for SAAC which in this case is the average objective function value of the Model-SAA. In other words, SAAC includes the total cost of setup and inventory plus the average cost incurred for overtime over all sample scenarios in K. Results in Table 2 indicate that the final optimality gap and the required computation time increase as the number of scenarios increases. Moreover, we observe the SAA heuristic with $|K| = 60$ obtains the smallest total expected cost over all instances compared to the other levels of $|K|$. The improvement provided by the SAA heuristic with $|K| = 60$ compared to the total expected cost of our baseline heuristic (see Table 1) is 3.12% on average.

Table 2: Results of the SAA method with $R = 1$, $\alpha = 1$, $\lambda = 1$ and $\rho = 50$

	SAA solutions				Evaluation		
Model-SAA	Opt	SAAC	Gap(%)	Seconds	EO	EOC	TEC
$\|K\| = 10$	354	64481.93	0.38	678.79	5.12	489.97	64940.26
$\|K\| = 20$	344	64548.40	0.40	699.67	3.12	310.57	64791.12
$\|K\| = 30$	340	64582.64	0.41	711.83	2.54	248.93	64747.45
$\|K\| = 40$	337	64604.68	0.42	723.05	2.24	213.73	64728.93
$\|K\| = 50$	336	64637.52	0.45	729.79	2.13	209.58	64743.66
$\boldsymbol{\|K\|=60}$	335	64635.31	0.44	732.35	2.06	205.39	**64726.41**
$\|K\| = 70$	334	64657.31	0.46	740.74	2.05	208.95	64739.29
$\|K\| = 80$	335	64667.18	0.47	749.58	1.92	190.47	64740.04
$\|K\| = 90$	333	64683.90	0.49	758.30	1.95	197.27	64749.30
$\|K\| = 100$	329	64689.36	0.50	763.26	1.95	202.44	64747.06

To further analyze the efficiency, we compare the total expected cost of the solutions obtained by the SAA heuristic with $|K| = 60$ to the total deterministic cost (including the cost of overtime) obtained by the Model-TR using the average setup times. Note that the latter value corresponds to a lower bound for the stochastic problem in case the Model-TR is solved to optimality. The explanation is as follows. For a given solution and a given realization of the setup times, the overtime is calculated as the maximum of zero and the total capacity used (i.e., total production time plus total realized setup times) minus the available capacity. This overtime calculation is hence a convex function of the setup time (since it is the maximum of two convex functions). Following Jensen's inequality (which indicates that a convex function applied to the average value of a random variable is always less than or equal to the average value of the convex function of the random variable), it holds that the overtime calculated using the average setup times (i.e., the overtime obtained in the deterministic solution when the setup times are set to their average value) is a lower bound on the average overtime with random setup times (i.e., the average value for the overtime when evaluated over the random setup times). If the Model-TR provides a suboptimal solution due to the imposed time limit, we use the final lower bound found by that formulation since it is also a valid lower bound for the stochastic problem (Model-SST). The average difference between the total expected cost of the solution (the total cost of setup, inventory and expected overtime) obtained by the SAA heuristic with $|K| = 60$ and this lower bound is 1.45%. In other words, we observe that the SAA heuristic with $|K| = 60$ provides very good solutions overall.

To further construct the SAA solutions, we experiment with various values of R, while fixing the size of K to 60. Specifically, we solve the test instances with 10, 20 and 40 replications. The corresponding results are given in Table 3. In this table, the second column (Opt) indicates the average number of instances (out of 540) solved to optimality over all the replications. The third column (LB) reports the statistical lower bound as calculated in Equation (37). The fourth column (UB) reports the best upper bound as calculated in Equation (38). The fifth column ($\frac{\text{UB-LB}}{\text{LB}}(\%)$) provides the information on the gap between the best upper and the statistical lower bound in percentage. The sixth column (Seconds) reports the average computation time in seconds per instance over all replications. The final column ($\hat{\sigma}_{\underline{z}}$) reports the standard deviation of the absolute gap as calculated in Equation (39).

Table 3: Results of the SAA method with 10, 20 and 40 replications with $|K| = 60$, $\alpha = 1$, $\lambda = 1$ and $\rho = 50$

Replication	Opt	LB	UB	$\frac{\text{UB-LB}}{\text{LB}}(\%)$	Seconds	$\hat{\sigma}_{\underline{z}}$
10	335.3	64247.4	64643.3	0.62	737.8	6.8
20	335.9	64264.2	64630.0	0.57	736.6	8.7
40	336.0	64273.9	64619.0	0.54	736.3	5.8

We observe that the statistical lower bound is increasing with the number of replications, whereas the upper bound is decreasing with the number of replications. The gap between these two values decreases hence with an increasing number of replications. The SAA method achieves an average gap of 0.54% using 40 replications. Note that a gap of 0.62% can be achieved with 10 replications, which consumes approximately a quarter of the computation time required for the 40 replications. The best upper bound found (64619.0) represents a 3.28% improvement over the upper bound achieved with the baseline heuristic (66811.2). The statistical lower bound obtained by 40 replications represents a 0.77% improvement compared to the lower bound calculated as the total deterministic cost (including the cost of overtime) obtained by the Model-TR using the average setup times. Finally, the average optimality gap reported by CPLEX [17] for the Model-SAA is 0.45%.

5.3. Heuristics H1 and H2: Changing Parameters in the Model-TR

In the baseline heuristic, we solved the Model-TR using the original values given for each problem parameter. We considered the values given by Trigeiro et al. [39] for each problem instance (no modification in the capacity or in setup times). In this new part of the computational experiments, we solve the Model-TR by employing a change in the parameter settings: (i) a smaller capacity for each period (H1) and (ii) larger setup times (H2).

The aim in (i) and (ii) is to introduce some buffer capacity in the deterministic solution, which leads to smaller expected overtime usage in the stochastic setting as explained at the end of Section 3.2. In case (i), we decrease the capacity by a given percentage, p. In case (ii), we set the setup times corresponding to a predetermined percentile of the setup time distribution. In other words, the setup times in model-TR are set equal to the w^{th} percentile of the corresponding Gamma distribution. The solutions obtained by employing the modified parameters in the deterministic model are then evaluated in the stochastic setting by considering the original values given for each problem element. More specifically, the stochastic evaluation is performed where the expected setup time of each item is equal to the original setup time and there is no reduction in the capacity.

The specific parameter settings used in the computational experiments are as follows: (i) a smaller capacity for each period (p = 0.5%, 1%, 1.5%, 2%, 2.5%, 3%) and (ii) larger setup times (w = 0.55, 0.60, 0.65, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95). Note that the Model-TR is always capable of ending up with a feasible solution by using overtime in case it is needed. Table 4 presents the solutions obtained by the Model-TR for the 540 instances. Note that in this table, SHC represents the total cost of setup and inventory. Results given in Table 4 indicate that the total expected cost decreases as we reduce the capacity up to a particular percentage, which is 1%. We observe that after this point, the total expected cost increases as the capacity further decreases. Similarly, the total expected cost decreases as we employ the setup times increased up to a particular percentile, which is the 75^{th} percentile. After this percentile, the total expected cost increases as the setup times further increase. The main reason behind these observations is as follows: as we decrease the capacity or increase the setup times at the deterministic planning level, we have more restricted resources compared to the one with the original parameters. This leads to higher setup and inventory holding costs, but also to a decrease in the expected overtime cost since we introduce buffer capacity. However, after a certain value (specifically $p = 1\%$ or $w = 0.75$), this is not the case anymore and the total expected costs start to increase.

Table 4: Results of H1 and H2 with $\alpha = 1$, $\lambda = 1$ and $\rho = 50$

	Deterministic Solution				Evaluation		
Model	Opt	SHC	Gap(%)	Seconds	EO	EOC	TEC
TR, p=0.005	368	64346.37	0.28	632.77	10.23	861.55	65207.91
TR, p=0.01	359	64596.09	0.33	657.73	3.40	265.49	64861.58
TR, p=0.015	356	64884.12	0.37	672.90	1.14	81.05	64965.18
TR, p=0.02	348	65213.48	0.43	684.76	0.46	40.61	65254.09
TR, p=0.025	344	65562.06	0.47	706.28	1.16	209.68	65771.76
TR, p=0.03	336	65949.80	0.60	732.40	6.32	1175.56	67125.36
TR, w=0.55	367	64175.45	0.26	616.76	25.57	2402.71	66578.15
TR, w=0.60	369	64266.54	0.27	625.78	12.34	1256.84	65523.38
TR, w=0.65	367	64364.32	0.30	637.39	6.97	760.46	65124.77
TR, w=0.70	361	64450.62	0.32	646.42	3.66	425.04	64875.67
TR, w=0.75	359	64576.45	0.35	656.27	1.68	210.71	**64787.18**
TR, w=0.80	355	64720.87	0.37	665.85	0.75	99.17	64820.04
TR, w=0.85	354	64907.31	0.42	681.58	0.32	43.77	64951.08
TR, w=0.90	339	65203.38	0.50	719.29	0.12	18.53	65221.92
TR, w=0.95	327	65635.77	0.62	763.16	0.56	99.23	65735.00

Results given in Table 4 show that changing problem parameters at the deterministic planning level yields production plans which perform better in the stochastic environment compared to those obtained by the Model-TR with original problem parameters (see Table 1). The solution obtained by the Model-TR with $w = 0.75$ (which provides the smallest total expected cost over all variants of the Model-TR) leads to an improvement in the total expected cost (compared to the baseline heuristic) of 3.03% on average, which is only slightly worse than the result obtained by the SAA heuristic. We further notice that the solutions obtained by Model-TR with w=0.70, 0.80 and 0.85 are similar to the ones obtained by the Model-TR with $w = 0.75$ (at most 0.25% difference in the total expected cost). We observe that the heuristic based on Model-TR with increased setup times yields more robust solutions compared to the heuristic based on the model with decreased capacity, which is more sensitive to changes in the value of p.

5.4. Effects of Increasing the Variability of the Gamma Distribution

In this section, we solve the 540 test instances by setting α to 0.0625 and λ to 16, leading to an increase in the coefficient of variation of the setup times by 4. For the SAA heuristic, we observe from Section 5.2 that the difference between the total expected costs obtained by considering a different number of scenarios is at most 0.33% on average. Therefore, we consider only three values for $|K|$, which are 20, 60 and 100, in this part of the experiments. Note that the increase in variation of setup times does not affect the Model-TR used in the baseline heuristic, nor the Model-TR with a smaller capacity used in H1 since the average setup times remain the same. The deterministic solutions obtained by these models (see Tables 1 and 4) are now evaluated with respect to the distribution with the new parameters ($\alpha = 0.0625$ and $\lambda = 16$). The Model-TR with modified setup times used in H2 and the Model-SAA are affected by the increase in variation since the probability distribution for the setup times has changed.

Tables 5, 6 and 7 present the corresponding results for the 540 test instances obtained by several methods (note that for each $|K|$ in Table 7, only one run is carried out). From Table 5, we observe that the results for the deterministic solution using the average setup times have not changed compared to the ones reported in Table 1, since the averages remain the same as discussed above. The stochastic evaluation of this solution leads to higher total expected costs. Comparing the results in Table 6 to those in Table 4, we also note for each model that the total expected cost increases as we increase the variation in setup times, due to the increase in the expected overtime cost. For H1, the results of the deterministic solution in Table 6 are the same as in Table 4, since the parameters used by that heuristic have not changed. The lowest total expected cost is achieved with a capacity reduction of 2.5%. We hence observe that with this increased variability in setup times, the capacity needs to be reduced more compared to the case with lower variability, where the best solution was obtained with a reduction of 1%. For H2, the results of the deterministic solution are different from those given in Table 4 (obtained by setting α to 1.00 and λ to 1.00) since the modified setup times used in that heuristic have changed due to the increase in variation. Moreover, there is a significant change in average computation times for the higher percentiles compared to those given in Table 4. The maximum average difference is 73.72% observed in the Model-TR employing the setup times modified with respect to the 95^{th} percentile. The total expected cost decreases as we employ the setup times increased up to the 75^{th} percentile. The improvement provided with this percentile compared to the total expected cost of our baseline heuristic (see Table 5) is 11.54% on average. After the 75^{th} percentile, the total expected cost increases as the setup times further increase. We observe that both for the case with regular variability and for the case with increased variability, the smallest total expected cost is achieved by modifying the setup times with respect to the 75^{th} percentile. From this experiment, it seems that the best choice of the percentile is not very sensitive to a big change in variability. We further notice that the total expected costs obtained by modifying the setup times with respect to 55^{th} percentile and 60^{th} percentile are higher than those obtained by the classical Model-TR. The reason behind this observation is as follows: when we have high variation in setup times, considering small percentiles underestimates the realizations of setup times. In other words, these two percentiles result in smaller setup times (compared to the original setup times which are equal to the expected setup times) leading to smaller regular costs (SHC), but larger expected overtime costs and larger total expected costs (compared to those obtained by the original problem elements).

In Table 7, the SAA heuristic with $|K| = 100$ provides the smallest total expected cost over all instances, where the improvement with respect to the total expected cost obtained by the naive baseline heuristic (the Model-TR) is 12.71% on average. Moreover, the solutions obtained by the SAA heuristic with $|K| = 20$ and with $|K| = 60$ lead to an improvement in the average total expected costs (compared to those obtained by the Model-TR) of 11.11% and 12.46%, respectively. In other words, the solutions obtained by the Model-SAA perform well in the stochastic environment when we consider a high variability in setup times.

To further analyze the effect of variability on the performance of the Model-SAA, we solve the 540 instances where R is set to 10, 20 and 40, and the size of K is equal to 60. The latter value is specifically chosen to be able to compare the solutions obtained by setting α to 0.0625 and λ to 16 to those obtained by setting α to 1.00 and λ to 1.00 (see Table 3). The corresponding results are given in Table 8. We observe for each R that the upper bound increases as we increase the variation in setup times, due to the increase in the expected overtime cost. The average number of instances solved to optimality decreases. Moreover, the average optimality gap reported by CPLEX over all the SAA problems (1.46%) increases compared to the gap reported over all the solutions given in Table 3 (0.45%). There is also a significant increase of 40.7% in average computation times compared to the results given in Table 3. Using the statistical lower bound provided by the SAA heuristic for $R = 10$, we see that the solutions obtained are within 5.52% of optimality. We observe that this optimality gap improves substantially when we perform 20 or 40 replications instead of just 10. When we compare the performance of the SAA method to that of the naive approach with respect to high variation in setup times, we observe that the best upper bound found by the SAA heuristic (68283.3) represents a 13.19% improvement over the upper bound achieved with the baseline heuristic (78658.23). The statistical lower bound obtained by 40 replications represents a 4.69% improvement compared to the lower bound calculated as the total deterministic cost (including the cost of overtime) obtained by the Model-TR using the average setup times. In other words, the SAA method performs well and provides significant improvements for the settings with high variation in setup times.

Table 5: Results of Model-TR with $\alpha = 0.0625$, $\lambda = 16$ and $\rho = 50$

	Deterministic Solution				Evaluation		
Model	Opt	TC	Gap(%)	Seconds	EO	EOC	TEC
TR	370	64154.80	0.25	613.99	162.85	14503.43	78658.23

Table 6: Results of H1 and H2 with $\alpha = 0.0625$, $\lambda = 16$ and $\rho = 50$

	Deterministic Solution				Evaluation		
Model	Opt	SHC	Gap(%)	Seconds	EO	EOC	TEC
TR, p=0.005	368	64346.37	0.28	632.77	132.30	11597.70	75944.06
TR, p=0.01	359	64596.09	0.33	657.73	106.49	9169.03	73765.11
TR, p=0.015	356	64884.12	0.37	672.90	86.28	7297.47	72181.58
TR, p=0.02	348	65213.48	0.43	684.76	69.53	5820.43	71033.92
TR, p=0.025	344	65562.06	0.47	706.28	56.44	4794.13	70356.21
TR, p=0.03	336	65949.80	0.60	732.40	50.73	4860.54	70810.33
TR, w=0.55	388	63831.36	0.20	558.86	256.72	21982.73	85814.11
TR, w=0.60	379	64018.01	0.24	595.86	186.33	16549.88	80567.89
TR, w=0.65	366	64346.67	0.31	634.09	116.07	10919.46	75266.12
TR, w=0.70	353	64872.39	0.43	690.89	65.07	6600.99	71473.37
TR, w=0.75	325	65511.92	0.69	776.92	36.40	4069.28	**69581.20**
TR, w=0.80	294	66172.13	0.87	866.75	26.88	3657.17	69829.31
TR, w=0.85	260	66952.01	0.94	993.51	57.19	9637.45	76589.47
TR, w=0.90	215	68134.71	1.23	1127.22	119.22	20864.38	88999.08
TR, w=0.95	160	70217.04	2.08	1325.79	264.61	45040.00	115257.04

Table 7: Results of the SAA method with $R = 1$, $\alpha = 0.0625$, $\lambda = 16$ and $\rho = 50$

	Deterministic Solution				Evaluation		
Model	Opt	SAAC	Gap(%)	Seconds	EO	EOC	TEC
SAA, $\|K\| = 20$	286	65707.02	1.09	915.37	40.31	4210.88	69917.88
SAA, $\|K\| = 60$	251	65966.45	1.45	1027.05	25.94	2892.58	68859.04
SAA, $\|K\|$=100	245	66043.28	1.61	1058.94	22.58	2619.63	**68662.90**

Table 8: Results of the SAA method with 10, 20 and 40 replications with $|K| = 60$, $\alpha = 0.0625$, $\lambda = 16$ and $\rho = 50$

Replication	Opt	LB	UB	$\frac{\text{UB-LB}}{\text{LB}}$(%)	Seconds	$\hat{\sigma}_{\underline{z}}$
10	246.5	64821.4	68401.3	5.52	1039.6	43.7
20	247.9	65992.1	68345.2	3.57	1037.0	32.1
40	292.6	65996.5	68283.3	3.47	1034.7	22.3

5.5. Effects of the Unit Overtime Cost

In this section, we evaluate the solutions obtained by the heuristic methods where α and λ are equal to 1.00, and the value of ρ is increased by a factor of 10 ($\rho = 500$). The cost of overtime is thus increased by a factor of 10.

Tables 9, 10 and 11 present the corresponding results for 540 instances obtained by several methods. Table 9 provides the result for our baseline heuristic. The total expected costs increase significantly, compared to those given in Table 1, since we increased the overtime cost. Comparing the results in Table 10 to those in Table 4, we also observe for each model that the total expected cost increases as the unit overtime cost increases. However, there is only a very small change in average computation times compared to those obtained by setting ρ to 50: the maximum average difference is less then 2%. For H1, results given in this table indicate that the total expected cost decreases as we reduce the capacity up to 2%. We observe that after this point, the total expected cost increases as the capacity further decreases. We hence notice that with the increased overtime costs, the capacity needs to be reduced more compared to the case with lower overtime costs. For H2, the total expected cost decreases as we employ the setup times increased up to the 85^{th} percentile. After this percentile, the total expected cost increases as the setup times further increase. The Model-TR with setup times modified according to the 85^{th} percentile provides the smallest total expected cost over all instances, where the improvement with respect to the total expected cost obtained by the baseline heuristic is 27.63% on average. The percentile level that leads to the lowest total expected costs has increased compared to the previous two cases. However, we also observe that the solutions obtained by the Model-TR with $w = 0.80$ and 0.90 are very similar to those obtained by the Model-TR with $w = 0.85$ (at most 0.54% difference in the total expected cost). In other words, the solutions obtained by the Model-TR with increased setup times perform well in the stochastic environment when we employ a high unit overtime cost. From Table 11, it is observed that the SAA heuristic with $|K| = 60$ and with $|K| = 100$ also perform well, where the average improvement with respect to the total expected cost obtained by the Model-TR is 27.38% on average and 27.61% on average, respectively.

To further analyze the effect of the unit overtime cost on the performance of the SAA heuristic, we solve 540 instances where R is set to 10, 20 and 40, and the size of K is equal to 60. The corresponding results are given in Table 12. Comparing the solutions given in this table to those in Table 3, we observe for each R that the upper bound increases as the unit overtime cost increases, due to the increase in the expected overtime cost. There is a slight decrease in the average number of instances solved to optimality. Moreover, the average optimality gap over all the SAA problems reported by CPLEX [17] is 0.59% (compared to 0.45% for the case with low overtime costs). We also observe a small change in the average computation times compared to those obtained by setting ρ to 50 (the maximum average difference is 1.11% when $R = 40$). Using the statistical lower bound provided by the SAA heuristic for $R = 10$, we see that the solutions obtained are within 1.52% of optimality. We observe that this optimality gap slightly improves when we perform 20 or 40 replications instead of just 10. When we compare the performance of the SAA method to that of the naive approach for the case with high unit overtime costs, we observe that the best upper bound found by the Model-SAA (65169.5) represents a 27.85% improvement over the upper bound achieved with the baseline heuristic (90330.63). The statistical lower bound obtained by 40 replications represents a 0.96% improvement compared to the lower bound calculated as the total deterministic cost (including the cost of overtime) obtained by the Model-TR using the average setup times.

Table 9: Results of Model-TR with $\alpha = 1$, $\lambda = 1$ and $\rho = 500$

	Deterministic Solution				Evaluation		
Model	Opt	TC	Gap(%)	Seconds	EO	EOC	TEC
TR	368	64157.47	0.25	615.17	28.98	26173.16	90330.63

Table 10: Results of H1 and H2 with $\alpha = 1$, $\lambda = 1$ and $\rho = 500$

	Deterministic Solution				Evaluation		
Model	Opt	SHC	Gap(%)	Seconds	EO	EOC	TEC
TR, p=0.005	366	64363.79	0.30	633.53	10.20	8567.33	72931.11
TR, p=0.01	361	64613.77	0.34	651.24	3.39	2633.26	67247.03
TR, p=0.015	356	64906.27	0.39	671.28	1.14	819.30	65725.57
TR, p=0.02	347	65268.16	0.58	687.48	0.45	388.39	65656.55
TR, p=0.025	344	65667.09	0.75	702.93	1.21	2171.07	67838.17
TR, p=0.03	335	66092.53	1.22	733.17	5.18	9557.40	75649.94
TR, w=0.55	368	64174.25	0.26	620.15	25.36	23643.53	87817.77
TR, w=0.60	370	64287.54	0.29	623.05	12.32	12493.40	76780.94
TR, w=0.65	363	64360.88	0.30	638.36	6.97	7597.91	71958.77
TR, w=0.70	362	64452.03	0.32	644.50	3.66	4260.27	68712.30
TR, w=0.75	360	64579.03	0.35	649.85	1.68	2105.39	66684.42
TR, w=0.80	356	64740.83	0.39	665.71	0.74	982.99	65723.82
TR, w=0.85	353	64936.08	0.43	679.94	0.31	432.83	**65368.91**
TR, w=0.90	345	65234.84	0.73	706.67	0.10	147.44	65382.30
TR, w=0.95	327	65703.66	1.37	762.49	0.81	1409.65	67113.31

Table 11: Results of the SAA method with $R = 1$, $\alpha = 1$, $\lambda = 1$ and $\rho = 500$

	Deterministic Solution				Evaluation		
Model	Opt	SAAC	Gap(%)	Seconds	EO	EOC	TEC
SAA, $\|K\| = 20$	344	64657.13	0.48	698.50	2.40	2270.52	66927.65
SAA, $\|K\| = 60$	336	64791.63	0.59	739.69	0.86	803.35	65594.98
SAA, $\|K\| = 100$	328	64866.84	0.66	774.17	0.55	528.21	65395.04

6. Conclusions

In this paper, we introduced the CLSP-SST which is a capacitated lot sizing problem with stochastic setup times. We described a mathematical

Table 12: Results of the SAA method with 10, 20 and 40 replications with $|K| = 60$, $\alpha = 1$, $\lambda = 1$ and $\rho = 500$

Replication	OPT	LB	UB	$\frac{\text{UB-LB}}{\text{LB}}$(%)	Runtime	$\hat{\sigma}_{\underline{z}}$
10	333.9	64273.1	65248.9	1.52	744.6	9.3
20	334.3	64335.2	65212.7	1.36	744.0	6.5
40	334.9	64335.4	65169.5	1.30	744.5	4.7

formulation that minimizes the total expected cost including regular costs and expected overtime costs. The expected overtime in any period is a convex function of the capacity consumption for that period in the deterministic counterpart. We have developed a procedure to effectively evaluate the expected overtime for setup times which follow a Gamma distribution, and proposed an SAA approach and two heuristics to obtain efficient production plans. Computational results showed that the SAA method provides very good solutions to be employed in stochastic settings. In the first case with low variability and low overtime costs, the SAA heuristic provides a 3.28% improvement compared to a naive heuristic. When we have high variability in setup times or when exceeding capacity brings a high violation cost, the improvement is 13.19% and 27.85% respectively. The SAA method also calculates a statistical lower bound and the results indicate that the solutions found are on average within 0.54% of this lower bound for the case with low variability and low overtime costs, within 3.47% for the case with high variability and within 1.30% for the case with high overtime costs.

In general, we have observed that heuristic H2, which is based on the deterministic model with overtime where setup times are modified with respect to the probability distribution, also performs very well. Setting the setup times equal to the 80^{th} percentile provides a very good overall compromise for the three cases we analysed (i.e., the case with low variability and low overtime costs, the case with high setup time variability, and the case with high overtime costs) and leads to high quality solutions which deviate only 1.14% on average from the best upper bounds for the various cases that we explored. This heuristic has the advantage that it is very easy to implement in practice, since we only have to solve an adapted deterministic model. Future research should focus on considering both stochastic setup times and stochastic processing times.

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