Perturbations for vibration of nano-beams of local/nonlocal mixture

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Abstract. Here we extend the perturbation approach, previously presented in the literature for Eringen's two-phase local/nonlocal mixture model, to free vibration of purely flexible beams. In particular, we expand the eigenvalues and the eigenvectors into power series of the fraction coefficient of the non-local material response up to 2nd order. We show that the family of 0th order bending couples satisfy the natural and essential boundary conditions of the 1st order; hence, the 1st order solution can conveniently be constructed using the eigenspace of the 0th order with no necessity of additional conditions. We obtain the condition of solvability that provides the incremental eigenvalue in closed form. We further demonstrate that the 1st order increment of the eigenvalue is always negative, providing the well-known softening effect of long-range interactions among the material points of a continuum modelled with Eringen's theory. We examine a simply supported beam as a benchmark problem and present the incremental eigenvalues in closed form.

Introduction

Structures with comparable internal and external length scales can be modelled by suitable quasi-continuum models, for which the distances shorter than their scale parameters have no physical meaning. Among many well-established models, we focus on Eringen's strain driven model, due to the efforts of Eringen and co-workers [1,2]. The application of this model to finite domains, however, needs additional mathematical conditions (the meaning of which is dubious) for a solution to exist in a certain form [3]. This led to criticisms on the validity of the material model itself, despite its strong mathematical and philosophical foundations [4].

Here we use the perturbation approach proposed in [5] to investigate free vibration of purely flexible beams composed of a local/nonlocal mixture. We get a hierarchy of equations that at the 0th order match with the well-known ones of local elastic beams. We show that the family of 0th order bending couples satisfy the natural and essential boundary conditions of 1st order; hence, the 1st order solution can conveniently be built using the 0th order eigenspace with no need of additional conditions. Exploiting the eigenfunctions orthogonality, we obtain a condition of solvability that provides the incremental eigenvalues in closed form and proves that the 1st order increment of the eigenvalue is always negative, yielding the well-known softening effect of long-range actions among the material points of a continuum according to Eringen's theory.

Direct 1-D Beam Model

We fix an origin and a Cartesian coordinate frame xyz, equipped with ortho-normal base vectors $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, in the 3D Euclidean ambient space. The reference configuration of a beam is defined as a collection of equal plane cross-sections attached through their centroid to a portion of the z-axis of length l, called beam axis. Another configuration is described by the translation of the cross-sections centroids, represented by the vector field $\mathbf{u}(z)$; and by the cross-sections rotation, represented by the orthogonal tensor field $\mathbf{R}(z)$. A suitable difference between these two configurations provides the following strain measures

$$\mathbf{E} = \frac{d\mathbf{R}}{dz}\mathbf{R}^{\mathrm{T}}, \qquad \mathbf{e} = \frac{d}{dz}(z\mathbf{k} + \mathbf{u}) - \mathbf{R}\mathbf{k}, \tag{1}$$

where $\mathbf{E}$ is a skew-symmetric tensor field of the curvature of the beam in the actual configuration and $\mathbf{e}$ provides shearing and elongation of the beam axis.

We are interested in planar motion; that is, the forces and couples acting on the beam are in the yz plane, and the deformed beam axis is also in the yz plane. Therefore, the components w, v of $\mathbf{u}(z)$ along z and y respectively are nonzero, and the only nonzero rotation angle Ω of the cross-sections is about the x-axis. Usual assumptions of small-amplitude displacements and rotations lead to following linearized strain measures

$$\varepsilon = \frac{dw}{dz}, \qquad \gamma = \frac{dv}{dz} + \Omega, \qquad \chi = \frac{d\Omega}{dz}, \tag{2}$$

where ε, γ, and χ stand for axial elongation, shearing strain, and bending curvature, respectively, while the displacement components and rotation now describe their first-order increments.

The balance in the actual shape, in the absence of the so-called *nonlocal residuals*, read

$$\frac{d\mathbf{n}}{dz} + \mathbf{q} = \mathbf{0}, \qquad \frac{d\mathbf{m}}{dz} + \frac{d(z\mathbf{k} + \mathbf{u})}{dz} \times \mathbf{n} + \mathbf{t} = \mathbf{0}, \tag{3}$$

where $\mathbf{n}$,$\mathbf{m}$ are inner force and couple, while $\mathbf{q}$,$\mathbf{t}$ stand for the external force and couple densities, respectively. When the couples and forces lie in the yz plane, Eq. (3) lead to the following scalar incremental balance equations

$$\frac{dN}{dz} + q_z = 0, \qquad \frac{dT}{dz} + q_y = 0, \qquad \frac{dM}{dz} - T + t_x = 0, \tag{4}$$

where: N, T, M are the increments of the inner normal force, shear force, and bending couple, respectively; q_z, q_y, t_x are the increments of the external actions reduced to the beam axis in the direction indicated by the subscripts.

The linear elastic problem is closed by the constitutive equation, which relates strain measures and dual work-conjugate internal actions. Here we assume that the beam is purely flexible, and the material obeys Eringen's model of a two-phase local/nonlocal mixture:

$$M = B[(1 - \xi)\chi + \xi K * \chi], \tag{5}$$

where B is the bending stiffness of the cross-section and ξ is the nonlocal portion of the material response; the latter is modelled by the kernel function K, accounting for long-range interactions among material points. We consider an exponential kernel in the following form

$$K * f = \int_0^L K(\zeta, z) f(\zeta) dz, \qquad K(\zeta, z) = \frac{1}{2\kappa} e^{\frac{|z-\zeta|}{\kappa}}, \qquad \int_{-\infty}^{\infty} K(\zeta, z) dz = 1 \tag{6}$$

where κ is called nonlocal parameter and is a rough measure of a 'radius of activity' or 'radius of extinction' of long-range interactions.

Transverse Vibration

Let the only non-zero component of external action be transverse inertia, that is,

$$q_z = t_x = 0, \qquad q_y = -m\ddot{v}, \tag{7}$$

m being the mass of the beam per unit length of its axis and over-dots denote time derivatives. If the fields of interest are harmonic in time with angular frequency ω, we choose to use the same notation to indicate their spatial part only, and define the following nondimensional quantities:

$$\bar{z} = \frac{z}{L}, \qquad \bar{v} = \frac{v}{L}, \qquad \bar{\kappa} = \frac{\kappa}{L}, \qquad \bar{T} = \frac{TL^2}{B}, \qquad \bar{M} = \frac{ML}{B}, \qquad \bar{\lambda} = \frac{m\omega^2 L^2}{B}. \tag{8}$$

For the ease of notation, the overbars will be omitted, except when confusion may arise.

With the assumptions on the initial and current shape of the beam, the transverse and axial motions are uncoupled. We are interested only in the transverse motion that is quantified by: the transverse displacement component v; the cross-section rotation angle Ω; its dual work conjugate, the bending couple M; and, for balance, the shear force T. The corresponding four 1st order integral-differential equations may be reduced into a single one of 4th order after trivial operations

$$(1-\xi)f^{IV}+\xi\left(K*f^{(i)}\right)^{(4-i)}-\lambda f=0,\quad \begin{matrix} f=v\Rightarrow i=1, & f=T\Rightarrow i=3, \\ f=\Omega\Rightarrow i=2, & f=M\Rightarrow i=4; \end{matrix} \tag{9}$$

where the superscript in the parentheses indicates the order of the spatial derivative.

The usual boundary conditions are listed in Table 1.

Table 1. *Natural and essential boundary conditions for particular selection of f.*

	pin	clamp	free
v	$f=0,$ $(1-\xi)f''+\xi K*f''=0$	$f=0,$ $f'=0$	$(1-\xi)f''+\xi K*f''=0$ $(1-\xi)f'''+\xi(K*f'')'=0$
Ω	$(1-\xi)f'+\xi K*f'=0$ $(1-\xi)f'''+\xi(K*f')''=0$	$(1-\xi)f'''+\xi(K*f')''=0$ $f=0$	$(1-\xi)f'+\xi K*f'=0$ $(1-\xi)f''+\xi(K*f')'=0$
T	$f'=0$ $(1-\xi)f'''+\xi K*f'''=0$	$f'=0$ $f''=0$	$f=0$ $(1-\xi)f'''+\xi K*f'''=0$
M	$f=0$ $f''=0$	$f''=0$ $f'''=0$	$f=0$ $f'=0$

It is notable that when the bending couple is used as the unknown field in Eq. 9, none of the classical boundary conditions include the convolution.

Perturbation with respect to the Nonlocal Fraction

If, as physically reasonable, all the quantities of interest depend on the nonlocal fraction ξ of the material response, we can approximate them by their ξ-power series expansions about a given value ξ_0

$$f\cong\sum_{j=0}^{n}\frac{(\xi-\xi_0)^j}{j!}f_j,\qquad \lambda\cong\sum_{j=0}^{n}\frac{(\xi-\xi_0)^j}{j!}\lambda_j,\qquad f_j=\left.\frac{\partial f}{\partial\xi}\right|_{\xi=\xi_0}. \tag{10}$$

A crucial choice for our aim is $\xi_0=0$, corresponding to a purely local elastic response. Thus, all quantities of interest are evaluated as if the beam were local. The expansions in Eq. 10 are fully reliable for 'small' values of ξ, i.e., $\xi\cong 0.1$, but are expected to give satisfactory results also for 'moderate' values of ξ, i.e., $\xi\cong 0.3-0.5$; in our applications we will give a numerical example.

Inserting Eq. 10 into Eq. 9 provides a hierarchy of equations for different orders of ξ:

0th order:
$$f_0^{IV}-\lambda_0 f_0=0, \tag{11}$$

1st order:
$$f_1^{IV}-\lambda_0 f_1=f_0^{IV}-\left(K*f_0^{(i)}\right)^{(4-i)}+\lambda_1 f_0, \tag{12}$$

2nd order:
$$f_2^{IV}-\lambda_0 f_2=2f_1^{IV}-2\left(K*f_1^{(i)}\right)^{(4-i)}+2\lambda_1 f_1+\lambda_2 f_0, \tag{13}$$

or, in general,

$$Df_0=0,\qquad Df_k=b_k,\qquad k=1,2,3,\ldots \tag{14}$$

for which the usual pattern of perturbation expansions is apparent: that is, the differential operator $D(\cdot)=d^4(\cdot)/dz^4-\lambda_0(\cdot)$ is the same at all orders and the 'forcing' terms on the right side depend on the solutions of previous orders. It is crucial to remark is that we turned the integral-differential system into a set of differential equations by perturbing the unknown field and the eigenvalues about the local problem, a solution of which we know to exist and be unique.

The boundary conditions for different ξ-orders are provided in Tables 2-3.

Table 2. *0^{th} order boundary conditions for particular selections of f.*

	pin	**clamp**	**free**
v	$f_0 = 0,$ $f_0'' = 0$	$f_0 = 0,$ $f_0' = 0$	$f_0'' = 0$ $f_0''' = 0$
Ω	$f_0' = 0$ $f''' = 0$	$f_0''' = 0$ $f_0 = 0$	$f_0' = 0$ $f_0'' = 0$
T	$f_0' = 0$ $f_0''' = 0$	$f_0' = 0$ $f_0'' = 0$	$f_0 = 0$ $f_0''' = 0$
M	$f_0 = 0$ $f_0'' = 0$	$f_0'' = 0$ $f_0''' = 0$	$f_0 = 0$ $f_0' = 0$

Table 3. *1^{st} order boundary conditions for particular selections of f.*

	pin	**clamp**	**free**
v	$f_1 = 0,$ $f_1'' = f_0'' - K * f_0''$	$f_1 = 0,$ $f_1' = 0$	$f_1'' = f_0'' - K * f_0''$ $f_1''' = f_0''' - \left(K * f_0''\right)'$
Ω	$f_1' = f_0' - K * f_0'$ $f_1''' = f_0''' - \left(K * f_0'\right)''$	$f_1''' = f_0''' - \left(K * f_0'\right)''$ $f_1 = 0$	$f_1' = f_0' - K * f_0'$ $f_1'' = f_0'' - \left(K * f_0'\right)'$
T	$f_1' = 0$ $f_1''' = f_0''' - K * f_0'''$	$f_1' = 0$ $f_1'' = 0$	$f_1 = 0$ $f_1''' = f_0''' - K * f_0'''$
M	$f_1 = 0$ $f_1'' = 0$	$f_1'' = 0$ $f_1''' = 0$	$f_1 = 0$ $f_1' = 0$

We see that the 1st order equations admit nonhomogeneous boundary conditions even though the corresponding 0th order equations have homogeneous boundary conditions.

Constructing a Solution

We define the scalar product for continuously differentiable functions h, g that satisfy the 0th order boundary conditions and have support (0,1) as

$$\langle h, g\rangle = \int_0^1 h(z)g(z)dz \tag{15}$$

The differential operator D is self-adjoint with respect to the scalar product defined in Eq. 15,

$$\langle Dh, g\rangle = \langle h, Dg\rangle. \tag{16}$$

We can also show that the family f_{0i} of solutions to Eq. 11, associated with the eigenvalue λ_{0i}, are orthogonal to each other; that is,

$$\langle f_{0i}, f_{0j}\rangle = 0, \qquad i \neq j. \tag{17}$$

Recalling that the boundary conditions for higher-order equations are homogeneous only if $f = M$, we choose the bending couple as unknown field for convenience in mathematical operations. Indeed, it is possible to write the bending couple by the following eigenfunction expansion

$$M_{ki} = M_{0i} + \sum\nolimits_{i\neq j} \alpha_{ij} M_{0j}\,, \qquad k = 1,2,3, \dots \tag{18}$$

for which the boundary conditions are satisfied; α_{ij} are the constants providing the contribution of the *j*th mode onto the *i*th mode of the 0th order solution.

Incremental Eigenvalues

Inserting Eq. 18 into Eq. 12 provides

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$$\sum_{i\neq j}\alpha_{ij}\lambda_{0j}M_{0j}-\lambda_{0i}\sum_{i\neq j}\alpha_{ij}M_{0j}=\lambda_{0i}M_{0i}-\lambda_{0i}K*M_{0i}+\lambda_{1i}M_{0i} \quad (19)$$

multiplying both sides by M_{0i} and integrating over the domain leads to

$$\int_0^1(\lambda_{0i}M_{0i}M_{0i}-\lambda_{0i}(K*M_{0i})M_{0i}+\lambda_{1i}M_{0i}M_{0i})dz=\langle M_{0i},b_1\rangle=0, \quad (20)$$

that is, the 0th order solution for the i^{th} mode shall be orthogonal to the forcing term of the 1st order equation for the corresponding mode. This is the Fredholm compatibility (solvability) condition, which we can solve for the incremental eigenvalue

$$\lambda_{1i}=\left(\frac{\langle K*M_{0i},M_{0i}\rangle}{\langle M_{0i},M_{0i}\rangle}-1\right)\lambda_{0i} \quad (21)$$

For a bounded and integrable function g with a compact support S the Lp norm is

$$\|g\|_p=\left(\int_S|g|^p dS\right)^{1/p}. \quad (22)$$

Young's Convolution Identity reads

$$\|f*g\|_r\leq\|f\|_p\|g\|_q, \qquad 1/p+1/q=1/r+1. \quad (23)$$

Then, we can write

$$\|K*M_{0i}\|_2\leq\|K\|_1\|M_{0i}\|_2,. \quad (24)$$

Since the kernel function K is a positive symmetric radial function, and considering Eq. 6-3, it is

$$\int_S K(\zeta,z)dz=\int_S|K(\zeta,z)|dz; \qquad \int_{-\infty}^{\infty}K(\zeta,z)dz=1\Rightarrow\int_0^1K(\zeta,z)dz=\|K\|_1\leq1, \quad (25)$$

which, along with Eq. 24, leads to

$$\|K*M_{0i}\|_2\leq\|M_{0i}\|_2,. \quad (26)$$

Referring to Hölder's Inequality

$$|\langle f*g\rangle|\leq\|f\|_2\|g\|_2, \quad (27)$$

we write

$$\int_0^1M_{0i}(K*M_{0i})dz\leq\left|\int_0^1M_{0i}(K*M_{0i})dz\right|\leq\left(\int_0^1M_{0i}^2dz\right)^{1/2}\left(\int_0^1|K*M_{0i}|^2dz\right)^{1/2}, \quad (28)$$

which, along with Eq. 26, gives

$$\int_0^1M_{0i}(K*M_{0i})dz\leq\int_0^1M_{0i}M_{0i}dz\Rightarrow\frac{\int_0^1M_{0i}(K*M_{0i})dz}{\int_0^1M_{0i}M_{0i}dz}=\frac{\langle K*M_{0i},M_{0i}\rangle}{\langle M_{0i},M_{0i}\rangle}\leq1. \quad (29)$$

This is a notable result indicating that the incremental eigenvalue is always negative, which is a feature ascribable to the well-known softening behavior of Eringen's model.

An Example: Simply supported beam

For a simply supported beam, the well-known local (0th order) solution reads

$$M_{0i}=C\sin i\pi z\,, \qquad \lambda_{0i}=i^4\pi^4 \quad (31)$$

where C is found by a suitable normalization.

Inserting Eq. 30 into Eq. 21 provides the slope of the $\lambda-\xi$ curve at $\xi=0$, that is, a linear approximation of the dependance of the eigenvalues on the non-local fraction coefficient. Further, what is most interesting is that such approximation is given in a closed form as follows

$$\lambda_{1i} = -\frac{\pi^6\kappa^2 i^6\{1+\kappa[i^2\pi^2\kappa + (-1)^i 2e^{-1/\kappa} - 2]\}}{(1+i^2\pi^2\kappa^2)^2} \tag{32}$$

Table 4. *Comparison of eigenvalues for different parameters.*

	$\kappa = 0.05$		$\kappa = 0.1$	
	$\xi = 0.25$	$\xi = 0.5$	$\xi = 0.25$	$\xi = 0.5$
[5]	9.84255	9.81485	9.77714	9.67978
[6]	9.84255	9.81485	9.77714	9.67978
Present	9.84276	9.81584	9.77854	9.68661

It is apparent in Table 4 that our perturbation approach can reply almost exactly the results found in the literature. It must be remarked that our linear perturbation approach is reliable even for moderate non-local fractions, which is not to be taken for granted in advance. That is, even though perturbations are usually reliable only for 'small' values of the perturbation parameter, in this case, as advanced previously, our perturbation yields reliable results also for 'non-small' values of ξ, i.e., $\xi = 0.5$.

Conclusions

By a perturbation approach we turned the integral-differential field system for beams composed of a two-phase local/nonlocal mixture into a hierarchy of bulk equations completed by the usual boundary conditions of local elasticity. Non-triviality of solutions and solvability conditions at successive steps of the hierarchy led to closed-form solutions for incremental natural angular frequencies of transverse natural vibration. Using well-known basic identities and inequalities of functional analysis we showed that the incremental eigenvalue is always negative, providing the well-known softening effect of Eringen's theory. The closed-form expressions can be of interest in the modelling and identification of nanomaterials.

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