

Paper Title: Investigating Fifth-Grade Students' Conceptions of Fractions on the Number Line  
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## Introduction

Fractions is one of the most difficult concepts for students in the elementary school mathematics curriculum (Charalambous & Pitta-Pantazi, 2007). It is widely acknowledged that many students have difficulties in translating among different representations of fractions, ordering fractions, and/or operating on fractions (Lamon, 2007). For example, 60% of United States 4<sup>th</sup> grade students could not identify whether  $1/4$  or  $1/5$  is larger. This situation was evident even after students received many years of instruction and that 50% of 8<sup>th</sup> grade students failed to order three fractions properly (National Achievement of Educational Progress Test (NAEP), 2007). In a similar vein, Turkish students who participated in international exams (i.e., Trends in International Mathematics and Science Study (TIMSS), 2011; Program for International Student Assessment (PISA), 2006) performed very weak especially in problems involving fractions compared to their counterparts from other countries (Yıldırım et. al., 2013).

Although approaches to teaching fractions and fraction's role in the curriculum could be different depending on the countries, using representations and manipulatives is common when the goal is conceptual understanding of fractions (Reys, Suydam, Lindquist, Smith, 1998). Among those, circular representations (i.e., pie, pizza etc.), rectangular shapes (i.e., chocolate, garden etc.), or discrete quantities are the frequently used instructional tools in order to facilitate student learning (Tunc-Pekkan, 2015). In recent years, use of number lines in teaching fractions has particularly been encouraged (National Math Advisory Board, 2008, p. 28) in the following manner: "One key mechanism linking conceptual and procedural knowledge is the ability to represent fractions on a number line." The number line representation may also reinforce the idea that fractions are numbers, which is difficult for many students to grasp (Stafylidou &

Vosniadou, 2004). Drawing on this assertion, the present study focused on students' conceptions of number lines in relation to their general fractional knowledge.

## **Literature Review**

### **Conceptual Framework related to Fraction knowledge**

Fractions have five different interpretations in the literature (English, 1995; Kieren, 1976): 1) part-whole interpretation (i.e.,  $\frac{3}{4}$  is represented as three parts out of four equal parts), 2) quotient meaning (i.e.,  $\frac{3}{4}$  is represented as 3 divided by 4), 3) operator construct (i.e.  $\frac{3}{4}$  is represented as three quarters of a number), 4) ratio (i.e.,  $\frac{3}{4}$  is represented as three parts to four parts), and 5) measure subconstruct (i.e.,  $\frac{3}{4}$  is represented as a point on the number line between 0 and 1).

In a comparative three-year study, Lamon (2007) investigated influence of teaching fractions focusing on different sub-constructs at a time on students' understanding. The results revealed that the measure sub-construct, which focuses on units, subintervals, and density of rational numbers, supported student understanding of fractions in better ways compared to instruction focused on other sub-constructs. Although one of the most suitable ways of assessing student understanding of measure is to use number line representations (Keijzer & Terwel, 2003), identifying fractions on the number line was also found to be one of the most difficult tasks for children (Charalambous & Pitta-Pantazi, 2007; Hannula, 2003; Kurt & Çakıroğlu, 2009; Tunç-Pekkan, 2015). Simply put, studying student fraction conceptions by focusing on the measure subconstruct and using number line representation is compelling.

In addition to measure subconstruct being related to number line representation, using operator construct to teach fractions has a great potential for students' conceptualization of fractions as numbers. While measure subconstruct is considered as extension of the part-whole

conceptions (Olive & Vomvoridi, 2006), measure subconstruct view also supports the view of operator subconstruct (Norton & Wilkins, 2010). Operator subconstruct entails activities with equipartitioning of a whole and producing new smaller units, and then iterating those units to check against the whole. In the literature, students' such conceptions are described as *partitive unit fraction scheme* (Steffe, 2002). For instance, when producing  $1/5$ , students can partition the given whole into 5 (equal) parts and then use one of those parts to iterate 5 times which will result as the whole. The students need to check whether iterating the quantity of  $1/5$  actually results in the original whole. It is important that 5 times  $1/5$ ,  $5/5$  and 1 are all equivalent in student's conception. In this paper, we are investigating how attempting to develop such conceptions during classroom instruction could result in students' use of measure subconstruct on the number line.

### **Student Understanding of Fractions on the Number Line**

A common student difficulty in ordering and locating fractions on the number line is the "whole number dominance". *Whole number dominance* referred to understanding of students who: "make separate comparisons of numerators and denominators using the ordering of whole numbers" (Behr, Wachsmuth, Post, and Lesh, 1984, p. 332) In contrast, students who are able to represent fractions on a number line and compare and order fractions are considered as "proficient multiplicative thinkers" (Stephens & Pearn, 2003).

Some researchers investigated reasons for student difficulties in understanding number line representations and identified skills, which could be related to understanding of number line representations. For instance, the results of a study by Hannula (2003), who investigated middle school students' conceptions of number line, revealed that a majority of students found it difficult to locate simple fractions on the number line; because of their lack in grasping the

meaning of the “whole” and conceptualizing a fraction as a number. Clearly, locating fractions on the number line task and shading a fraction of a bar task appeared to be requisites for each other, indicating that there is a relationship between students’ competencies in those skills. On the other hand, Smith, Solomon and Carey (2005, p. 121) found that students who “had a clear division model of fraction notation (i.e., numbers can be repetitively divided and the result would not yield to zero) were able to order fractions correctly. That is, students’ conceptual understanding of fractions involved an “inter-related body of representations including representations of division and density of number.” Similarly, Hartnett and Gelman (1998) argued that there should be a conceptual shift in students’ minds in order for them to understand fractions as numbers. The number concept constructed in a young child’s mind is considered “incommensurable” (Smith et al., p. 133) with the fraction number concept and thus leads to difficulties in student understanding of fractions as a number.

Although previous research focused on understanding student thinking of fractions on the number line through their performance during tests (Charalambous & Pitta-Pantazi, 2007; Kurt & Çakıroğlu, 2009) and clinical interviews (Smith et al., 2005), there is a need for studies which take into account student learning in mathematics classrooms and the nature of student experiences in the learning of fractions, specifically in relation to number lines (Behr et al. 1992; Lamon, 2007). Students’ experiences in mathematics classroom should be taken into account to get an in depth understanding of their performance (Boaler, 1998; Henningsen & Stein, 1997; Stigler & Hiebert, 2009). Although it is difficult to establish a direct relationship between teaching and learning, we believe that unpacking both processes has the potential to understand how the teaching of fractions can be improved to enhance student learning.

The intervention implemented in this study was based on Fraction Scheme Theory developed by Steffe and Olive (2010), which focused on operator subconstruct understanding of fractions. In operator subconstruct, multiplicative relationship and units-coordination between the quantities are important which requires using unit fractions as the smallest measurement unit to name other quantities. An example of a problem situation asking students to engage in multiplicative reasoning and incorporate number lines was the following in our intervention: “If the size of your chocolate bar is 1 and mine is 10 times bigger than yours, show my chocolate bar with Cuisinera rods and draw it on your notebook. Think about the reverse situation: If the size of my chocolate bar is ‘1’ what would you call yours? Using Cuisinera rods as measurement units,  $\frac{1}{10}$  as the unit, can you draw a number line where you could show ‘1’?”. We hypothesized that operator subconstruct activities would enable students perform better with measure subconstruct problem situations especially when locating numbers on number line and their activities related to fractions as numbers. In this accordance, our two research questions were: (1) How do fifth-grade students make sense of fractions represented on number lines in the context of fraction schemes?; and (2) How do fifth-grade students make sense of fractions represented on number lines in the context of measurement subconstruct of fractions? More specifically, our aim was to investigate student conceptions of fractions specific to number lines in the context of instruction based on operator and measurement sub-constructs. This study has a potential to help mathematics teachers and teacher educators better understand students’ thinking about fractions. This, in turn, may inform improved design of instruction, which will lead to a more effective learning of fractions.

## **Methods**

### **Context for the Larger Study**

This paper reports results from a larger research project involving a university-school partnership where teacher educators (researchers) acted as teachers in a disadvantaged urban school for a year. The researchers aimed to not only improve students' mathematics achievement with a particular focus on fractional knowledge but also investigated processes involved in becoming teacher-researchers within such partnership.

Four teacher-researchers planned and taught lessons based on the Fraction Scheme Theory (Steffe & Olive, 2010) and also followed the fifth-grade mathematics curriculum (Ministry of National Education, 2013) covering unit, simple, compound (improper), and equivalent fractions; location and ordering fractions on the number line; and addition and subtraction of fractions. Specifically, instruction focused on understanding how to operate on fractions as measurement units. Different than the centralized curriculum, the research team designed and taught learning activities using virtual manipulatives (Java Bars, Figure 1) and Cuisenaire rods (Figure 2), both of which have a potential to help students understand measurement sub construct of fractions in more effective ways.

In order to assess the development in students' fractional knowledge acquisition, the researchers developed a 32-item multiple-choice Fractions Test (FT) based on the literature on students' difficulties and misconceptions on fractions. The FT was administered to experimental students who received University within School-based instruction and control students who received traditional instruction as a pre- and post-test. The results of the analysis of variance tests showed that the teaching of fractions through university-school partnership integrated into the fifth-grade mathematics classes promoted students' gains in fractional knowledge ( $F(1, 216) = 4.420, p = .03, \eta^2 = .49$ ). Specifically, among the intervention students ( $N = 112$ ) there was a 45.74% improvement in the mean scores from pretest ( $M = 10.32, SD = 3.88$ ) to posttest ( $M =$

15.73  $SD= 6.21$ ) whereas among the non-intervention students ( $N=108$ ), there was 30% improvement in the mean scores from pretest ( $M= 12.10$   $SD= 4.87$ ) to posttest ( $M= 15.73$   $SD= 6.21$ ). For details of this intervention study readers are referred to Aydın, Birgili, Tunç-Pekkan, Taylan, & Özcan (2016).

### **Clinical Interviews**

Drawing on aforementioned findings, this study focused on qualitative aspects of growth in student learning in fractions, in the context of number line as they were instructed with a particular focus on the measurement and operator sub-constructs. The research team selected six students from the experimental group based on their overall mathematics achievements (2 low-, 2 middle- and 2 high-achievers). The selection procedure relied upon both mathematics teacher decisions (i.e., views on student class participation) and students' prior achievement in mathematics (i.e., written exam grades). Students were interviewed individually before and after learning the fractions topic. The names of the participants are pseudonyms: Gabriel, Frank (low-achievers); Emily, Iris (middle-achievers) and Erin, Sam (high-achievers). Two teacher-researchers who were trained in conducting clinical interviews (Ginsburg, 1997; Goldin, 2000) completed 6 pre-instruction and 6 post-instruction clinical interviews. Both pre- and post-interviews were composed of the same five questions that focused on the measurement and operator subconstructs of fractions. The 20-45 min-interviews were videotaped and student work was scanned. Both verbal and nonverbal student responses were transcribed in detail. In line with our research questions, we were interested in understanding how the intervention influenced students' conceptions of fractions on a number line in a qualitative manner.

The interview questions were developed and analyzed according to significant fraction concepts mentioned in previous literature (Bright, Behr, Post, & Wachsmuth, 1988; Lamon,

1999; Smith et al. 2005; Tunç-Pekkan, 2015). As research team analyzed and compared pre- and post-instruction clinical interviews among students at different levels of fractional understanding, the contrasting performance of students in two tasks about the number line were of particular value. Henceforth, student thinking with regards to the two tasks (see Figure 3 and Figure 4).

### **Task 1( Figure 3)**

### **Task 2 (Figure 4 )**

### **Results**

The analyses of the comparisons of pre-and post-instruction clinical interviews confirmed that there was substantial improvement in students' conceptual understanding of measurement subconstruct of fractions which was also evident in their performance in the FT. Comparing students' performance in Task 1 (see Figure 3) and Task 2 (see Figure 4), which were both related to assessing students' conceptions of fractions in the context of number lines, revealed an interesting contrast in students' performance which required further attention.

Task 2 in the clinical interview protocol required students to locate 1 precisely on the number line given given  $\frac{5}{9}$  and 0. It was important that students would be able to understand finding unit fraction and pay attention to equal length intervals to find whole (1). Although this task was considered a difficult task in the literature (Hannula, 2003; Lamon, 1999) and required reversible thinking, four of six students in our study improved greatly in making sense of this task after the intervention.

Below is a typical example where a low-achieving student, Gabriel improved her understanding of this task during the post-instruction clinical interview compared to the pre-instruction clinical interview. As indicated by Figure 5, in the pre-instruction clinical interview, Gabriel located  $\frac{1}{1}$  next to 0 because she reasoned: "1 comes after 0." She seemed to think

about ordering whole numbers instead of fractions. She did not consider  $5/9$  in the problem situation.

During the post-instruction interview, however, Gabriel was able to partition the interval 0 to  $5/9$  correctly and with precision (2 squares between each sub-interval) and located 1 on the number line correctly (see Figure 6). She knew  $9/9$  was equal to 1. This was a big improvement for her. In general, students demonstrated similar improvements as Gabriel did in this task.

#### **Insert Figure 6**

On the other hand, Task 1, which required students to construct a number line, locate benchmarks (0 and 1) and discuss density of the number line, still proved to be challenging for the students in our study during the post-instruction interviews. Only one student out of six demonstrated the desired understanding of density and infinity of fractional numbers and started to consider fractions as numbers in the post-instruction clinical interview (Figure 7): “Just like we divided the interval 0-1 in half, we can divide the interval 0 to  $1/2$  into half and we get  $1/4$ . We can keep dividing in this manner. Therefore there are many numbers in between 0 and 1.” (Erin, a high-achieving student, transcription from the post-instruction interview).

#### **Insert Figure 7**

Although both Task 1 and Task 2 have been considered as difficult to conceptualize for students in general, the students’ performance in the present study differed greatly in these tasks. An important pattern in students’ answers on Task 1 was the whole-number dominance in students’ thinking as discussed in previous literature (Behr et al., 1984). The transcripts provided below confirm previously made arguments on the incommensurability of the concept of number in the context of fractions (Smith et al., 2005). In other words, the students in our study seemed to have a difficulty in this task because they did not consider fractions as numbers in the

conventional sense even though it was emphasized during instruction. This is especially interesting considering that they performed better in the other number line task (Task 2). The patterns related to student understanding of fractions are provided in the following paragraphs.

### **Student Difficulties in Understanding Density of Fractions**

The incommensurability between the whole number and fraction concepts is evident during the following post-instruction interviews. The first excerpt is part of an interview conducted with a middle-achiever student, Iris. The second excerpt was conducted with a high-achieving student, Erin.

During the post clinical interview, Iris knew how to show a half and a quarter of an interval from 0 to 1 on a number line she constructed. She could also correctly compare the quantities of “quarter” and “half” and know how to write them in the fraction form. On the other hand, she did not seem to consider fractions as numbers. As evident in her answer, although she knew  $1/4$  was between 0 and  $1/2$  on the number line, she thought there were no numbers between 0 and  $1/2$ .

T (Teacher): OK, well can you mark the half of the interval of 0 (zero) and  $1/2$  (one over two)?

T (Teacher): What do you think about that number?

I:  $1/4$  (one over four) because if I take half one more time, it becomes a quarter.

T (Teacher): Does this become a quarter, hmm ok. Let's write what this is.

T (Teacher): Well is this number larger than  $1/2$  or smaller than  $1/2$ ?

I: Smaller.

T (Teacher): How did you understand?

I: Because this (pointing to  $1/2$ ) is larger than that.

T (Teacher): Is  $1/2$  larger than that?

I: Yes

T (Teacher): Why?

I: For example while drawing ... For example, think a cake. When you cut the cake in half, there are larger pieces. But when we divide it into four, the pieces are smaller.

T (Teacher): Hmm, ok. Is there any other number here, for instance is there any other number between 0 and  $\frac{1}{4}$ ? Or between 0 and  $\frac{1}{2}$ ?

I: Nothing

T (Teacher): Nothing?

I: No.

T (Teacher): Ok. You say that there isn't any number between 0 and  $\frac{1}{2}$ . Ok. So where is the place of  $\frac{1}{4}$ ? Is it between 0 and  $\frac{1}{2}$ ?

I: Yes.

T (Teacher): It is between them but there is not any number between 0 and  $\frac{1}{2}$ ?

I:  $\frac{1}{4}$  is between 0 and  $\frac{1}{2}$ ?

T (Teacher): There is  $\frac{1}{4}$ . Ok what is that between 0 and  $\frac{1}{4}$ ? Do you think that there are any numbers between them?

I: Nothing.

T (Teacher): Nothing?

I: No.

T (Teacher): Ok. Well, let's think about this: How many numbers are there between 0 and 1?

I: Between 0 and 1, there are five.

Similarly, the following high-achieving student, Erin, who had a clear division model in fractions, demonstrated hesitation in considering fractions as numbers in the following excerpt:

Teacher (T): Well, are there any numbers between 0 and  $1/2$ ?

E: “ only numbers like 1, 2, 3?”

T: It can be any number.

Only after getting this approval from the teacher, the student replies:

E: For example, we divided 1 by 2, we found  $1/2$ . Similarly we can find different numbers.

### **Student Difficulties in Partitioning of a Unit “1” on the Number Line**

The following student not only had a difficulty in understanding density of fractions but also demonstrated difficulties in partitioning of a whole on the number line during completion of Task 1. It is worthwhile to recognize that Emily, a middle-achiever demonstrated some understanding of the part-whole meaning of the fractions despite having difficulties in understanding the unit and was able to complete Task 2 correctly. The concept of unit, however, proved to be challenging for her.

Emily marked the interval of 5 unit squares starting from '0' in order to show the quarter of a number line of twenty squares because there were five squares within the quarter of the number line. Then she marked and wrote as  $1/5$  underneath. As evident in the figure below, she later changed  $1/5$  to  $1/2$  and had  $1/2$  in different locations on the same number line. The excerpt below demonstrated where her initial difficulty resulted from.

### **Insert Figure 8**

Teacher: Well can you mark the half of the interval of 0 and  $1/2$ ?

The student marks the half correctly.

T: How did you decide to mark there?

S: We divided 10 by 10. Half of 10, when divided by 2. It becomes 5. By counting 5 units, I found the half.

T: Can you say which number the half corresponds to?

S: If we do not assume here, (by showing other parts) it is  $1/2$ . Because if we divide 5 by 5.

T: Now this was  $1/2$ . I asked you the place you had just marked. What does it correspond to?

S:  $1/5$ .

T: Does it become  $1/5$  ? Why?

S: There are 5 spaces. It exists 2 parts. Two parts become  $1/5$  in case of starting from 1 at the beginning.

---

T: Is there any number between these places? Between 0 and 1.

S: I said  $1/5$ . If we divide it into other equal pieces, it becomes a number.

T: There is a number. There may be other numbers. Can you explain it to me how you were sure that this number is  $1/5$ . Why not  $1/6$ , for example, rather than  $1/5$ ?

S: Since here is 5 equal pieces. If there is 1 instead of  $1/2$ , here becomes  $1/2$ , not  $1/5$ .

T: However our 1 is here. And in this case, do you say that it becomes  $1/5$ ?

S: Yes.

T: Fine. I want you to estimate. How many numbers are there between 0 and 1? Can you count them?

S: There are twenty.

T: are there twenty? You might put twenty. But what are those numbers?

S: many numbers. This number line becomes 20. If we enlarge this number line, we can get another big number. Many numbers exist between them.

As evident from the above transcript and Figure 8, Emily's conception of '1' on the number line changed many times. The unit or the whole number 1 on the number line was not

fixed and it was not a number that was produced by iterations of different unit fractions. Unit fractions could be produced by partitioning of flexible wholes but they could not be used for producing fixed quantity of '1'. Therefore, Emily was not able to demonstrate multiplicative reasoning that produced number of '1'.

### **Design Decisions related to Conducting the Clinical Interviews**

We made several design decisions (Goldin, 2000) about conducting the clinical interviews. First, we asked students to show their work on a squared paper during the interviews. One advantage of using a squared paper was making sure that the students were not only estimating where to locate a fraction. Instead, our goal was to assess whether students were aware of a need to locate fractions in precise ways and determine units correctly. Although each student did take into account of squares on the paper when determining half of the length of the number line between 0 and 1, drawing on the squared paper revealed some misconceptions. The excerpt from Emily's post-interview (Figure 8) demonstrates an example to such misconceptions. It is probable that Emily confused 5 squares with  $\frac{1}{5}$ , instead of identifying the point correctly as  $\frac{1}{4}$ . Similarly, another student drew a number line with a length of 20 squares from 0 to 1, she identified half of the number line as the number  $\frac{1}{10}$  because she counted 10 squares as the half of the number line. From this perspective, using squared paper in the number line tasks may have reinforced the whole number bias (or dominance) the students already had. A second design decision was to ask the students to construct their own number lines. Although this aspect of the interview allowed us to assess the students' understanding of the number line concept in a more detailed manner, it also led to difficulties in comparing students' answers. For instance, some students had uneven numbered length of squares between 0 and 1 in their number lines and had a harder time in determining the half of the number line as opposed to using even

numbers of squares. The design decisions require a follow up with using number line tasks on plain paper and using standard number lines for each student.

### **Discussion**

The contrast of the same group of students' performance in two tasks revealed importance of nature of assessment tools in understanding students' thinking thoroughly. Constructing a number line and being aware of the density of the numbers may require higher-order thinking skills. Although students can locate "1" on a number line where 0 and  $\frac{5}{9}$  are given (this requires partitioning, iterating, and reversible thinking), understanding there are infinitely many rational numbers in between 0 and  $\frac{1}{2}$  requires recursive partitioning scheme and having a clear model of division (Smith et al., 2005; Steffe & Olive, 2010).

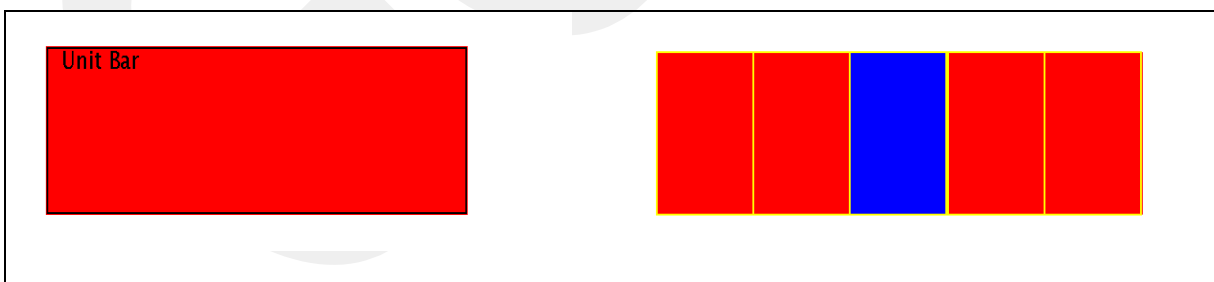
Because the authors of the paper were also teachers of the students, the analyses of student answers not only reveal strengths and weaknesses of student understanding, but also about teaching which shaped student understanding and constructing of the concept of fractions on the number line. The most frequently used tools in the introduction of the fraction concept were virtual manipulatives (Java bars) and Cuisinera rods, both of which helped to reinforce measurement aspects of fractions. While the number line items in the FT indicated that experimental students performed better than their counterparts in the control group, clinical interviews revealed students' difficulty in understanding the density in fractions and making unit coordination may persist in some students, which may possibly require further intervention.

Considering the students' general success in Task 2, it may be argued that locating fractions on the number line and understanding the measure subconstruct was possible with instruction emphasizing the operator subconstruct. The focus of instruction undertaken in this study may offer an explanation to the contrasting findings mentioned in the literature, which

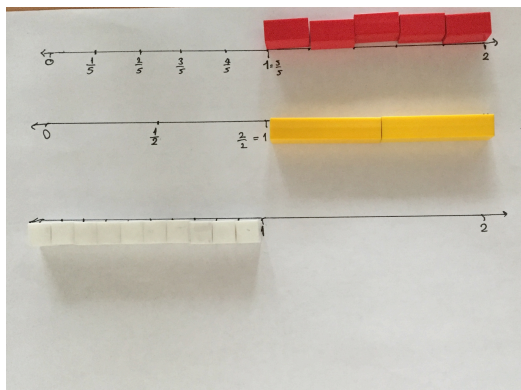
acknowledged the difficulty of tasks similar to Task 2. On the other hand, there is a need to explore other instructional approaches, which can lead to a successful transition from understanding the operator fractional knowledge to being competent with density of fractions (Task 1). Future research may find it worthwhile to investigate students' understanding of the density on the numberline and its connection to the measure and operator subconstructs of fractions.

### **Educational and Scientific Importance of the Research**

Results of this research can help advance the existing literature on how students understand fractions on the number line. As teachers strive to build instruction on students' thinking (Franke & Kazemi, 2001; Hiebert & Grouws, 2007), our findings have the potential to help teachers improve their teaching practices by taking into consideration the typical responses of fifth-grade students of different levels of understanding of fractions in general, and understanding of fractions represented on the number line, in particular. Additionally, analyzing student thinking by considering instruction focused on Fractions Scheme Theory has the potential to make connections between student thinking and instruction.



**Figure 1.** An example of a whole and its equal parts presented with JAVABARS.



**Figure 2. Instruction using Cuisinera rods.**

**Task 1.** Draw a number line and locate 0 and 1. Show middle of the interval 0-1. What is the number that corresponds to the middle of this interval? Are there any numbers between 0 and 1? If so, how many numbers are there?

**Figure 3. Task 1: Locating benchmarks on a number line.**

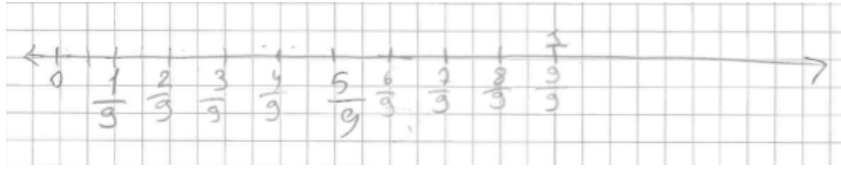
**Task 2:** The interviewer draws a number line where 0 and  $\frac{5}{9}$  are located 10 squares apart on the squared paper. Please locate 1 on this number line . Why did you locate 1 here?



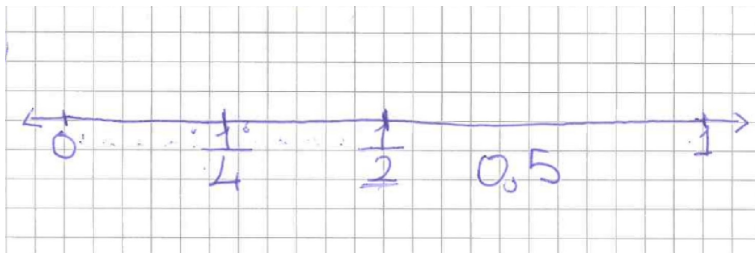
**Figure 4. Task 2: Given 0 and  $\frac{5}{9}$ , locating 1 on the number line.**



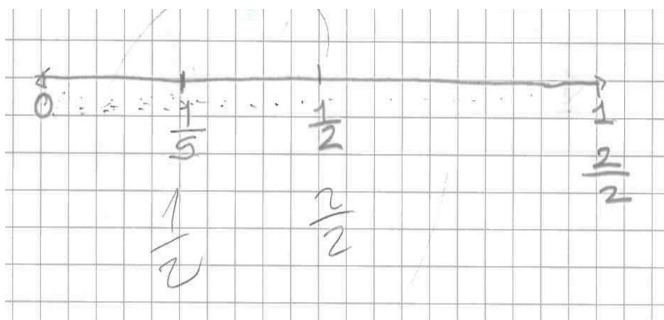
**Figure 5. Given 0 and  $\frac{5}{9}$ , locating 1 (pre-instruction interview, Gabriel).**



**Figure 6.** Given 0 and  $\frac{5}{9}$ , locating 1 (post-instruction interview, Gabriel).



**Figure 7.** Locating benchmarks on a number line (post-instruction interview, Erin).



**Figure 8.** Task 1 (Emily, post-instruction interview).

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